8 Graphing Quadratic Functions

Town Population ***(p. 464)***

Satellite Dish ***(p. 457)***

Roller Coaster ***(p. 448)***

Firework Explosion ***(p. 437)***

Garden Waterfalls ***(p. 430)***

Maintaining Mathematical Proficiency

Graphing Linear Equations

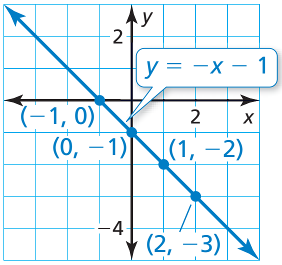
Example 1 **Graph $y = -x - 1$.**

Step 1 Make a table of values.

x	$y = -x - 1$	y	(x, y)
-1	$y = -(-1) - 1$	0	$(-1, 0)$
0	$y = -(0) - 1$	-1	$(0, -1)$
1	$y = -(1) - 1$	-2	$(1, -2)$
2	$y = -(2) - 1$	-3	$(2, -3)$

Step 2 Plot the ordered pairs.

Step 3 Draw a line through the points.



Graph the linear equation.

1. $y = 2x - 3$

2. $y = -3x + 4$

3. $y = -\frac{1}{2}x - 2$

4. $y = x + 5$

Evaluating Expressions

Example 2 **Evaluate $2x^2 + 3x - 5$ when $x = -1$.**

$$\begin{aligned} 2x^2 + 3x - 5 &= 2(-1)^2 + 3(-1) - 5 && \text{Substitute } -1 \text{ for } x. \\ &= 2(1) + 3(-1) - 5 && \text{Evaluate the power.} \\ &= 2 - 3 - 5 && \text{Multiply.} \\ &= -6 && \text{Subtract.} \end{aligned}$$

Evaluate the expression when $x = -2$.

5. $5x^2 - 9$

6. $3x^2 + x - 2$

7. $-x^2 + 4x + 1$

8. $x^2 + 8x + 5$

9. $-2x^2 - 4x + 3$

10. $-4x^2 + 2x - 6$

11. **ABSTRACT REASONING** Complete the table. Find a pattern in the differences of consecutive y-values. Use the pattern to write an expression for y when $x = 6$.

x	1	2	3	4	5
$y = ax^2$					

Mathematical Practices

Mathematically proficient students try special cases of the original problem to gain insight into its solution.

Problem-Solving Strategies

Core Concept

Trying Special Cases

When solving a problem in mathematics, it can be helpful to try special cases of the original problem. For instance, in this chapter, you will learn to graph a quadratic function of the form $f(x) = ax^2 + bx + c$. The problem-solving strategy used is to first graph quadratic functions of the form $f(x) = ax^2$. From there, you progress to other forms of quadratic functions.

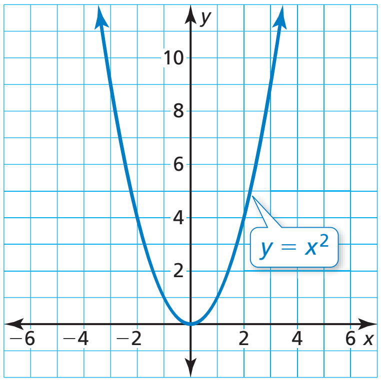
$f(x) = ax^2$	Section 8.1
$f(x) = ax^2 + c$	Section 8.2
$f(x) = ax^2 + bx + c$	Section 8.3
$f(x) = a(x - h)^2 + k$	Section 8.4

EXAMPLE 1 Graphing the Parent Quadratic Function

Graph the parent quadratic function $y = x^2$. Then describe its graph.

SOLUTION

The function is of the form $y = ax^2$, where $a = 1$. By plotting several points, you can see that the graph is U-shaped, as shown.



▶ The graph opens up, and the lowest point is at the origin.

Monitoring Progress

Graph the quadratic function. Then describe its graph.

1. $y = -x^2$

2. $y = 2x^2$

3. $f(x) = 2x^2 + 1$

4. $f(x) = 2x^2 - 1$

5. $f(x) = \frac{1}{2}x^2 + 4x + 3$

6. $f(x) = \frac{1}{2}x^2 - 4x + 3$

7. $y = -2(x + 1)^2 + 1$

8. $y = -2(x - 1)^2 + 1$

9. How are the graphs in Monitoring Progress Questions 1−8 similar? How are they different?

8.1 Graphing $f(x) = ax^2$

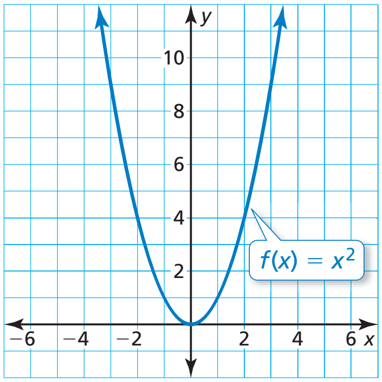
Essential Question

Essential Question What are some of the characteristics of the graph of a quadratic function of the form $f(x) = ax^2$?

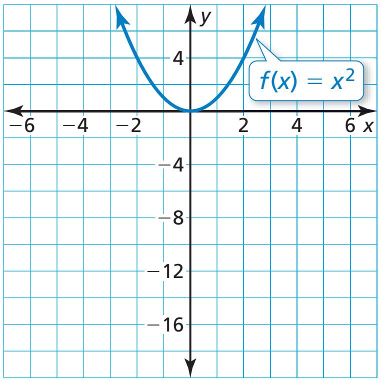
EXPLORATION 1 Graphing Quadratic Functions

Work with a partner. Graph each quadratic function. Compare each graph to the graph of $f(x) = x^2$.

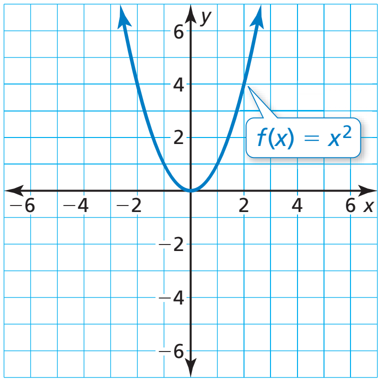
a. $g(x) = 3x^2$



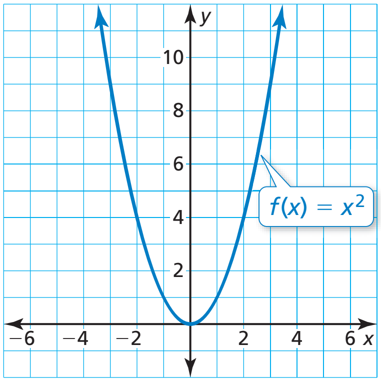
b. $g(x) = -5x^2$



c. $g(x) = -0.2x^2$



d. $g(x) = \frac{1}{10}x^2$



REASONING QUANTITATIVELY

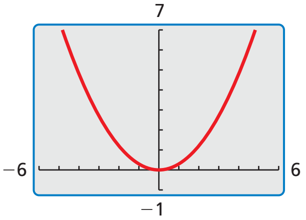
To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

Communicate Your Answer

2. What are some of the characteristics of the graph of a quadratic function of the form $f(x) = ax^2$?

3. How does the value of a affect the graph of $f(x) = ax^2$? Consider $0 < a < 1$, $a > 1$, $-1 < a < 0$, and $a < -1$. Use a graphing calculator to verify your answers.

4. The figure shows the graph of a quadratic function of the form $y = ax^2$. Which of the intervals in Question 3 describes the value of a? Explain your reasoning.



8.1 Lesson

Core Vocabulary

quadratic function, *p. 420*
parabola, *p. 420*
vertex, *p. 420*
axis of symmetry, *p. 420*

Previous
domain
range
vertical shrink
vertical stretch
reflection

What You Will Learn

- Identify characteristics of quadratic functions.
- Graph and use quadratic functions of the form $f(x) = ax^2$.

Identifying Characteristics of Quadratic Functions

A **quadratic function** is a nonlinear function that can be written in the standard form $y = ax^2 + bx + c$, where $a \neq 0$. The U-shaped graph of a quadratic function is called a **parabola**. In this lesson, you will graph quadratic functions, where b and c equal 0.

REMEMBER

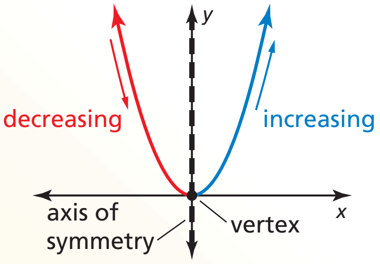
The notation $f(x)$ is another name for y.

Core Concept

Characteristics of Quadratic Functions

The *parent quadratic function* is $f(x) = x^2$. The graphs of all other quadratic functions are *transformations* of the graph of the parent quadratic function.

The lowest point on a parabola that opens up or the highest point on a parabola that opens down is the **vertex**. The vertex of the graph of $f(x) = x^2$ is (0, 0).

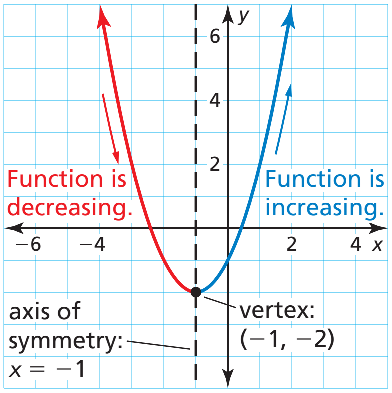


The vertical line that divides the parabola into two symmetric parts is the **axis of symmetry**. The axis of symmetry passes through the vertex. For the graph of $f(x) = x^2$, the axis of symmetry is the y-axis, or $x = 0$.

EXAMPLE 1 Identifying Characteristics of a Quadratic Function

Consider the graph of the quadratic function.

Using the graph, you can identify characteristics such as the vertex, axis of symmetry, and the behavior of the graph, as shown.



You can also determine the following:

- The domain is all real numbers.
- The range is all real numbers greater than or equal to −2.
- When $x < -1$, y increases as x decreases.
- When $x > -1$, y increases as x increases.

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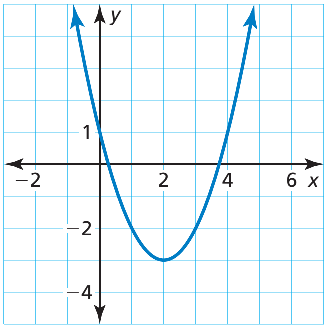
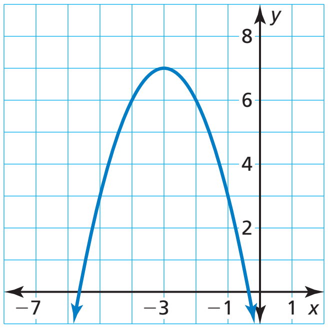
Identify characteristics of the quadratic function and its graph.

1.



2.



REMEMBER

The graph of $y = a \cdot f(x)$ is a vertical stretch or shrink by a factor of a of the graph of $y = f(x)$.

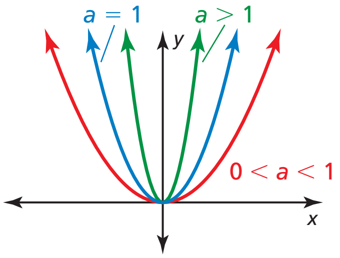
The graph of $y = -f(x)$ is a reflection in the x-axis of the graph of $y = f(x)$.

Graphing and Using $f(x) = ax^2$

Core Concept

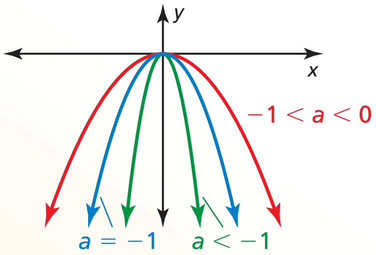
Graphing $f(x) = ax^2$ When $a > 0$

- When $0 < a < 1$, the graph of $f(x) = ax^2$ is a vertical shrink of the graph of $f(x) = x^2$.
- When $a > 1$, the graph of $f(x) = ax^2$ is a vertical stretch of the graph of $f(x) = x^2$.



Graphing $f(x) = ax^2$ When $a < 0$

- When $-1 < a < 0$, the graph of $f(x) = ax^2$ is a vertical shrink with a reflection in the x-axis of the graph of $f(x) = x^2$.
- When $a < -1$, the graph of $f(x) = ax^2$ is a vertical stretch with a reflection in the x-axis of the graph of $f(x) = x^2$.



EXAMPLE 2 Graphing $y = ax^2$ When $a > 0$

Graph $g(x) = 2x^2$. Compare the graph to the graph of $f(x) = x^2$.

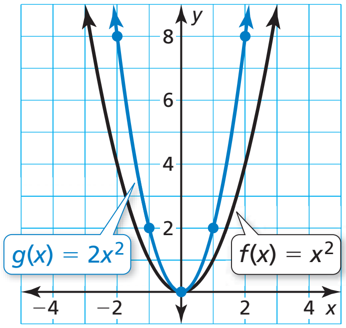
SOLUTION

Step 1 Make a table of values.

x	−2	−1	0	1	2
g(x)	8	2	0	2	8

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



▶ Both graphs open up and have the same vertex, (0, 0), and the same axis of symmetry, $x = 0$. The graph of g is narrower than the graph of f because the graph of g is a vertical stretch by a factor of 2 of the graph of f.

EXAMPLE 3 Graphing $y = ax^2$ When $a < 0$

Graph $h(x) = -\frac{1}{3}x^2$. Compare the graph to the graph of $f(x) = x^2$.

SOLUTION

STUDY TIP

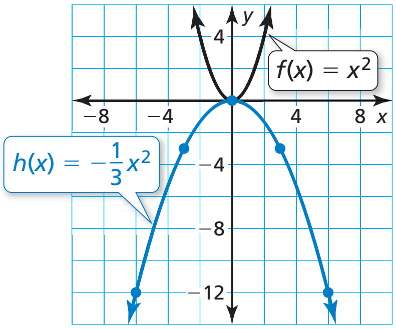
To make the calculations easier, choose x-values that are multiples of 3.

Step 1 Make a table of values.

x	-6	-3	0	3	6
$h(x)$	-12	-3	0	-3	-12

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



▶ The graphs have the same vertex, (0, 0), and the same axis of symmetry, $x = 0$, but the graph of h opens down and is wider than the graph of f. So, the graph of h is a vertical shrink by a factor of $\frac{1}{3}$ and a reflection in the x-axis of the graph of f.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

3. $g(x) = 5x^2$

4. $h(x) = \frac{1}{3}x^2$

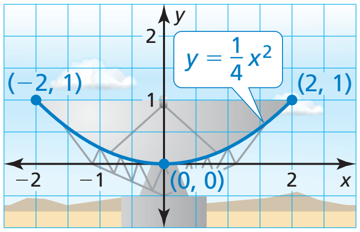
5. $n(x) = \frac{3}{2}x^2$

6. $p(x) = -3x^2$

7. $q(x) = -0.1x^2$

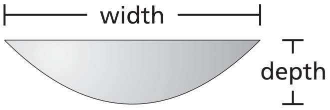
8. $g(x) = -\frac{1}{4}x^2$

EXAMPLE 4 Solving a Real-Life Problem



The diagram at the left shows the cross section of a satellite dish, where x and y are measured in meters. Find the width and depth of the dish.

SOLUTION



Use the domain of the function to find the width of the dish. Use the range to find the depth.

The leftmost point on the graph is $(-2, 1)$, and the rightmost point is $(2, 1)$. So, the domain is $-2 \le x \le 2$, which represents 4 meters.

The lowest point on the graph is $(0, 0)$, and the highest points on the graph are $(-2, 1)$ and $(2, 1)$. So, the range is $0 \le y \le 1$, which represents 1 meter.

▶ So, the satellite dish is 4 meters wide and 1 meter deep.

Monitoring Progress

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9. The cross section of a spotlight can be modeled by the graph of $y = 0.5x^2$, where x and y are measured in inches and $-2 \le x \le 2$. Find the width and depth of the spotlight.

8.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

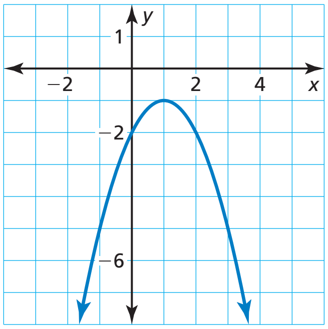
1. **VOCABULARY** What is the U-shaped graph of a quadratic function called?

2. **WRITING** When does the graph of a quadratic function open up? open down?

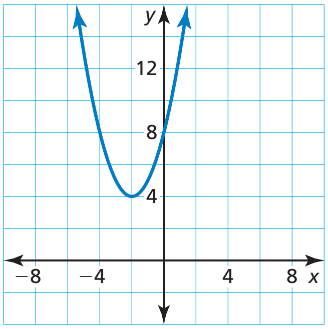
Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, identify characteristics of the quadratic function and its graph. *(See Example 1.)*

3.



4.



In Exercises 5–12, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Examples 2 and 3.)*

5. $g(x) = 6x^2$

6. $b(x) = 2.5x^2$

7. $h(x) = \frac{1}{4}x^2$

8. $j(x) = 0.75x^2$

9. $m(x) = -2x^2$

10. $q(x) = -\frac{9}{2}x^2$

11. $k(x) = -0.2x^2$

12. $p(x) = -\frac{2}{3}x^2$

In Exercises 13–16, use a graphing calculator to graph the function. Compare the graph to the graph of $y = -4x^2$.

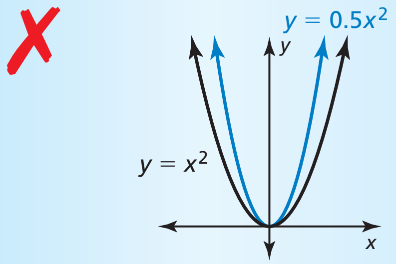
13. $y = 4x^2$

14. $y = -0.4x^2$

15. $y = -0.04x^2$

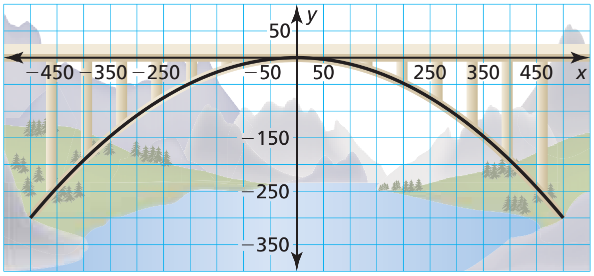
16. $y = -0.004x^2$

17. ERROR ANALYSIS Describe and correct the error in graphing and comparing $y = x^2$ and $y = 0.5x^2$.



The graphs have the same vertex and the same axis of symmetry. The graph of $y = 0.5x^2$ is narrower than the graph of $y = x^2$.

18. MODELING WITH MATHEMATICS The arch support of a bridge can be modeled by $y = -0.0012x^2$, where x and y are measured in feet. Find the height and width of the arch. *(See Example 4.)*



19. PROBLEM SOLVING The breaking strength z (in pounds) of a manila rope can be modeled by $z = 8900d^2$, where d is the diameter (in inches) of the rope.

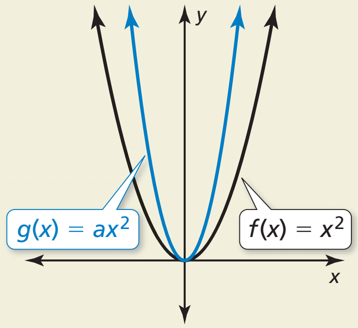
a. Describe the domain and range of the function.

b. Graph the function using the domain in part (a).

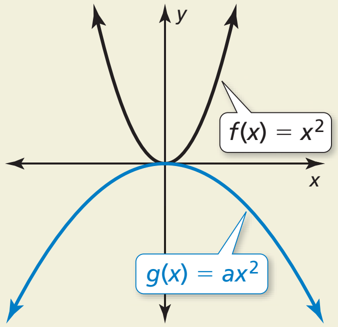
c. A manila rope has four times the breaking strength of another manila rope. Does the stronger rope have four times the diameter? Explain.

20. **HOW DO YOU SEE IT?** Describe the possible values of a.

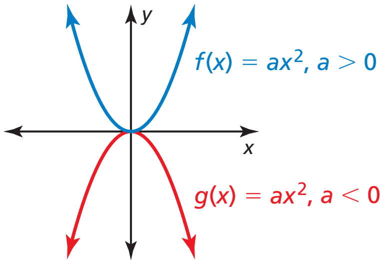
a.



b.



ANALYZING GRAPHS **In Exercises 21–23, use the graph.**



21. When is each function increasing?

22. When is each function decreasing?

23. Which function could include the point $(-2, 3)$? Find the value of a when the graph passes through $(-2, 3)$.

24. **REASONING** Is the x-intercept of the graph of $y = ax^2$ always 0? Justify your answer.

25. **REASONING** A parabola opens up and passes through $(-4, 2)$ and $(6, -3)$. How do you know that $(-4, 2)$ is not the vertex?

ABSTRACT REASONING **In Exercises 26–29, determine whether the statement is *always*, *sometimes*, or *never* true. Explain your reasoning.**

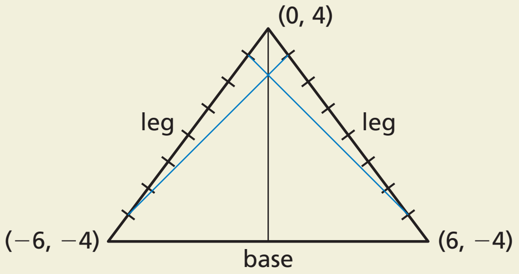
26. The graph of $f(x) = ax^2$ is narrower than the graph of $g(x) = x^2$ when $a > 0$.

27. The graph of $f(x) = ax^2$ is narrower than the graph of $g(x) = x^2$ when $|a| > 1$.

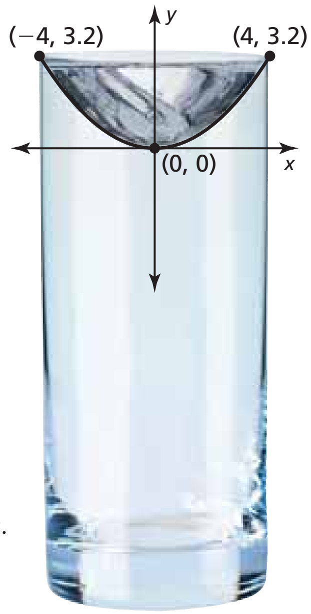
28. The graph of $f(x) = ax^2$ is wider than the graph of $g(x) = x^2$ when $0 < |a| < 1$.

29. The graph of $f(x) = ax^2$ is wider than the graph of $g(x) = dx^2$ when $|a| > |d|$.

30. **THOUGHT PROVOKING** Draw the isosceles triangle shown. Divide each leg into eight congruent segments. Connect the highest point of one leg with the lowest point of the other leg. Then connect the second highest point of one leg to the second lowest point of the other leg. Continue this process. Write a quadratic equation whose graph models the shape that appears.



31. **MAKING AN ARGUMENT** The diagram shows the parabolic cross section of a swirling glass of water, where x and y are measured in centimeters.



a. About how wide is the mouth of the glass?

b. Your friend claims that the rotational speed of the water would have to increase for the cross section to be modeled by $y = 0.1x^2$. Is your friend correct? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression when $n = 3$ and $x = -2$. *(Skills Review Handbook)*

32. $n^2 + 5$

33. $3x^2 - 9$

34. $-4n^2 + 11$

35. $n + 2x^2$

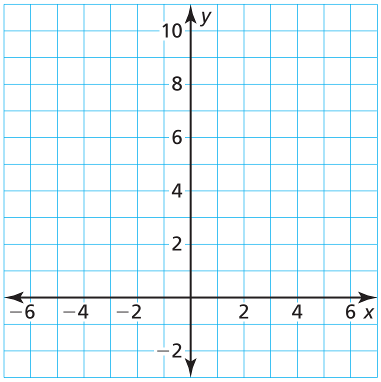
8.2 Graphing $f(x) = ax^2 + c$

Essential Question How does the value of c affect the graph of $f(x) = ax^2 + c$?

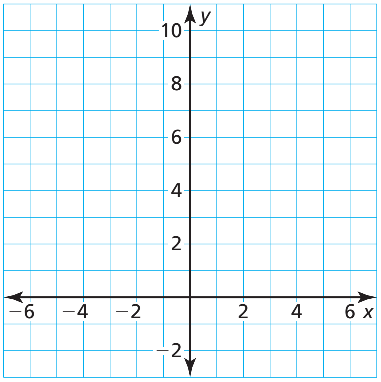
EXPLORATION 1 Graphing $y = ax^2 + c$

Work with a partner. Sketch the graphs of the functions in the same coordinate plane. What do you notice?

a. $f(x) = x^2$ and $g(x) = x^2 + 2$



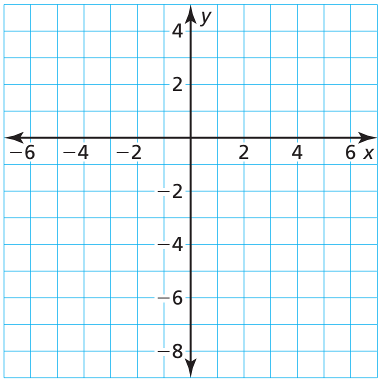
b. $f(x) = 2x^2$ and $g(x) = 2x^2 - 2$



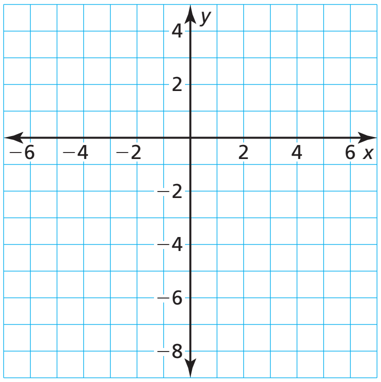
EXPLORATION 2 Finding x-Intercepts of Graphs

Work with a partner. Graph each function. Find the x-intercepts of the graph. Explain how you found the x-intercepts.

a. $y = x^2 - 7$



b. $y = -x^2 + 1$

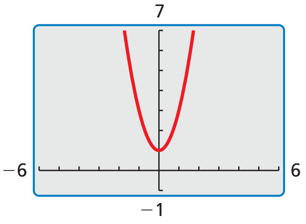


USING TOOLS STRATEGICALLY

To be proficient in math, you need to consider the available tools, such as a graphing calculator, when solving a mathematical problem.

Communicate Your Answer

3. How does the value of c affect the graph of $f(x) = ax^2 + c$?
4. Use a graphing calculator to verify your answers to Question 3.
5. The figure shows the graph of a quadratic function of the form $y = ax^2 + c$. Describe possible values of a and c. Explain your reasoning.



8.2 Lesson

What You Will Learn

- Graph quadratic functions of the form $f(x) = ax^2 + c$.
- Solve real-life problems involving functions of the form $f(x) = ax^2 + c$.

Core Vocabulary

zero of a function, *p. 428*

Previous

translation
vertex of a parabola
axis of symmetry
vertical stretch
vertical shrink

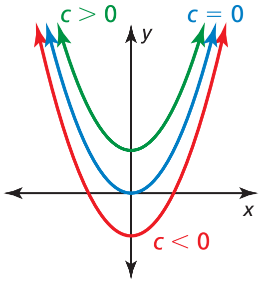
Graphing $f(x) = ax^2 + c$

Core Concept

Graphing $f(x) = ax^2 + c$

- When $c > 0$, the graph of $f(x) = ax^2 + c$ is a vertical translation c units up of the graph of $f(x) = ax^2$.
- When $c < 0$, the graph of $f(x) = ax^2 + c$ is a vertical translation $|c|$ units down of the graph of $f(x) = ax^2$.

The vertex of the graph of $f(x) = ax^2 + c$ is $(0, c)$, and the axis of symmetry is $x = 0$.



EXAMPLE 1 Graphing $y = x^2 + c$

Graph $g(x) = x^2 - 2$. Compare the graph to the graph of $f(x) = x^2$.

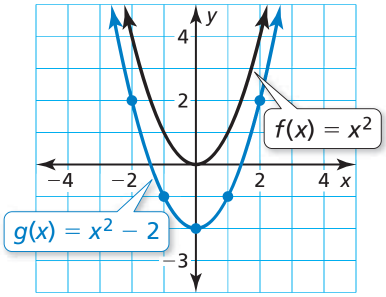
SOLUTION

Step 1 Make a table of values.

x	-2	-1	0	1	2
$g(x)$	2	-1	-2	-1	2

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



REMEMBER

The graph of $y = f(x) + k$ is a vertical translation, and the graph of $y = f(x - h)$ is a horizontal translation of the graph of f.

- Both graphs open up and have the same axis of symmetry, $x = 0$. The vertex of the graph of g, $(0, -2)$, is below the vertex of the graph of f, $(0, 0)$, because the graph of g is a vertical translation 2 units down of the graph of f.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

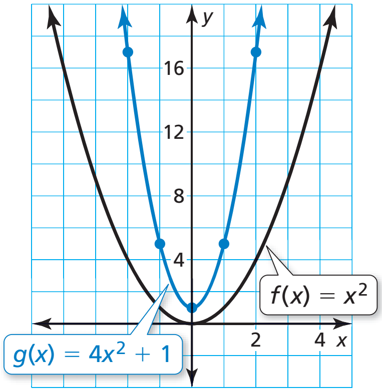
1. $g(x) = x^2 - 5$

2. $h(x) = x^2 + 3$

EXAMPLE 2 Graphing $y = ax^2 + c$

Graph $g(x) = 4x^2 + 1$. Compare the graph to the graph of $f(x) = x^2$.

SOLUTION



Step 1 Make a table of values.

x	−2	−1	0	1	2
$g(x)$	17	5	1	5	17

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

▶ Both graphs open up and have the same axis of symmetry, $x = 0$. The graph of g is narrower, and its vertex, (0, 1), is above the vertex of the graph of f, (0, 0). So, the graph of g is a vertical stretch by a factor of 4 and a vertical translation 1 unit up of the graph of f.

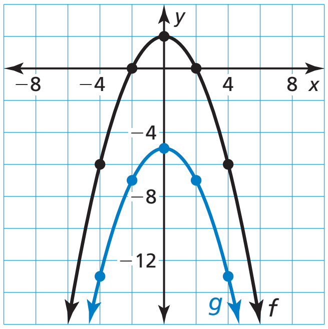
EXAMPLE 3 Translating the Graph of $y = ax^2 + c$

Let $f(x) = -0.5x^2 + 2$ and $g(x) = f(x) - 7$.

a. Describe the transformation from the graph of f to the graph of g. Then graph f and g in the same coordinate plane.

b. Write an equation that represents g in terms of x.

SOLUTION



a. The function g is of the form $y = f(x) + k$, where $k = -7$. So, the graph of g is a vertical translation 7 units down of the graph of f.

x	−4	−2	0	2	4
$f(x)$	−6	0	2	0	−6
$g(x)$	−13	−7	−5	−7	−13

$-0.5x^2 + 2$

$f(x) - 7$

b. $g(x) = f(x) - 7$ Write the function g.

$= -0.5x^2 + 2 - 7$ Substitute for $f(x)$.

$= -0.5x^2 - 5$ Subtract.

▶ So, the equation $g(x) = -0.5x^2 - 5$ represents g in terms of x.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

3. $g(x) = 2x^2 - 5$

4. $h(x) = -\frac{1}{4}x^2 + 4$

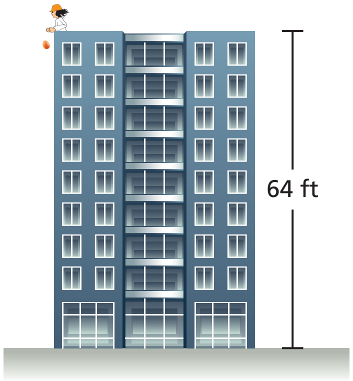
5. Let $f(x) = 3x^2 - 1$ and $g(x) = f(x) + 3$.

- **a.** Describe the transformation from the graph of f to the graph of g. Then graph f and g in the same coordinate plane.
- **b.** Write an equation that represents g in terms of x.

Solving Real-Life Problems

A **zero of a function** f is an x-value for which $f(x) = 0$. A zero of a function is an x-intercept of the graph of the function.

EXAMPLE 4 Solving a Real-Life Problem



The function $f(t) = -16t^2 + s_0$ represents the approximate height (in feet) of a falling object t seconds after it is dropped from an initial height s_0 (in feet). An egg is dropped from a height of 64 feet.

a. After how many seconds does the egg hit the ground?

b. Suppose the initial height is adjusted by k feet. How will this affect part (a)?

SOLUTION

1. **Understand the Problem** You know the function that models the height of a falling object and the initial height of an egg. You are asked to find how many seconds it takes the egg to hit the ground when dropped from the initial height. Then you need to describe how a change in the initial height affects how long it takes the egg to hit the ground.

2. **Make a Plan** Use the initial height to write a function that models the height of the egg. Use a table to graph the function. Find the zero(s) of the function to answer the question. Then explain how vertical translations of the graph affect the zero(s) of the function.

COMMON ERROR

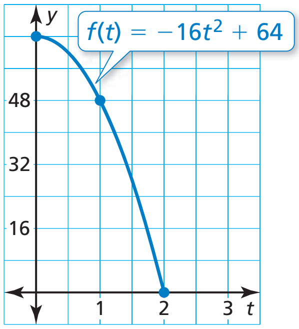
The graph in Step 1 shows the height of the object over time, not the path of the object.

3. **Solve the Problem**

a. The initial height is 64 feet. So, the function $f(t) = -16t^2 + 64$ represents the height of the egg t seconds after it is dropped. The egg hits the ground when $f(t) = 0$.

Step 1 Make a table of values and sketch the graph.

t	0	1	2
$f(t)$	64	48	0



Step 2 Find the positive zero of the function. When $t = 2$, $f(t) = 0$. So, the zero is 2.

▶ The egg hits the ground 2 seconds after it is dropped.

b. When the initial height is adjusted by k feet, the graph of f is translated up k units when $k > 0$ or down $|k|$ units when $k < 0$. So, the x-intercept of the graph of f will move right when $k > 0$ or left when $k < 0$.

▶ When $k > 0$, the egg will take more than 2 seconds to hit the ground. When $k < 0$, the egg will take less than 2 seconds to hit the ground.

4. **Look Back** To check that the egg hits the ground 2 seconds after it is dropped, you can solve $0 = -16t^2 + 64$ by factoring.

Monitoring Progress

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6. Explain why only nonnegative values of t are used in Example 4.
7. **WHAT IF?** The egg is dropped from a height of 100 feet. After how many seconds does the egg hit the ground?

8.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. VOCABULARY State the vertex and axis of symmetry of the graph of $y = ax^2 + c$.

2. WRITING How does the graph of $y = ax^2 + c$ compare to the graph of $y = ax^2$?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 1.)*

3. $g(x) = x^2 + 6$

4. $h(x) = x^2 + 8$

5. $p(x) = x^2 - 3$

6. $q(x) = x^2 - 1$

In Exercises 7–12, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 2.)*

7. $g(x) = -x^2 + 3$

8. $h(x) = -x^2 - 7$

9. $s(x) = 2x^2 - 4$

10. $t(x) = -3x^2 + 1$

11. $p(x) = -\frac{1}{3}x^2 - 2$

12. $q(x) = \frac{1}{2}x^2 + 6$

In Exercises 13–16, describe the transformation from the graph of f to the graph of g. Then graph f and g in the same coordinate plane. Write an equation that represents g in terms of x. *(See Example 3.)*

13. $f(x) = 3x^2 + 4$
$g(x) = f(x) + 2$

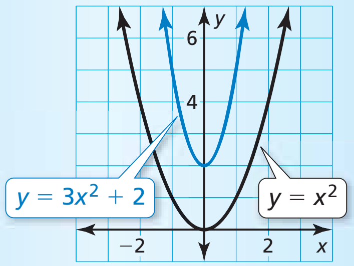
14. $f(x) = \frac{1}{2}x^2 + 1$
$g(x) = f(x) - 4$

15. $f(x) = -\frac{1}{4}x^2 - 6$
$g(x) = f(x) - 3$

16. $f(x) = 4x^2 - 5$
$g(x) = f(x) + 7$

17. ERROR ANALYSIS Describe and correct the error in comparing the graphs.

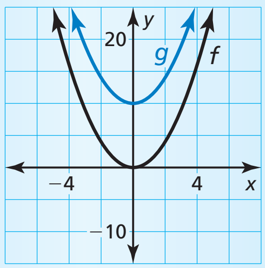
✗



The graph of $y = 3x^2 + 2$ is a vertical shrink by a factor of 3 and a translation 2 units up of the graph of $y = x^2$.

18. ERROR ANALYSIS Describe and correct the error in graphing and comparing $f(x) = x^2$ and $g(x) = x^2 - 10$.

✗



Both graphs open up and have the same axis of symmetry. However, the vertex of the graph of g, (0, 10), is 10 units above the vertex of the graph of f, (0, 0).

In Exercises 19–26, find the zeros of the function.

19. $y = x^2 - 1$

20. $y = x^2 - 36$

21. $f(x) = -x^2 + 25$

22. $f(x) = -x^2 + 49$

23. $f(x) = 4x^2 - 16$

24. $f(x) = 3x^2 - 27$

25. $f(x) = -12x^2 + 3$

26. $f(x) = -8x^2 + 98$

27. MODELING WITH MATHEMATICS A water balloon is dropped from a height of 144 feet. *(See Example 4.)*

- **a.** After how many seconds does the water balloon hit the ground?
- **b.** Suppose the initial height is adjusted by k feet. How does this affect part (a)?

28. MODELING WITH MATHEMATICS The function $y = -16x^2 + 36$ represents the height y (in feet) of an apple x seconds after falling from a tree. Find and interpret the x- and y-intercepts.

In Exercises 29–32, sketch a parabola with the given characteristics.

29. The parabola opens up, and the vertex is (0, 3).

30. The vertex is (0, 4), and one of the x-intercepts is 2.

31. The related function is increasing when $x < 0$, and the zeros are -1 and 1.

32. The highest point on the parabola is $(0, -5)$.

33. **DRAWING CONCLUSIONS** You and your friend both drop a ball at the same time. The function $h(x) = -16x^2 + 256$ represents the height (in feet) of your ball after x seconds. The function $g(x) = -16x^2 + 300$ represents the height (in feet) of your friend's ball after x seconds.

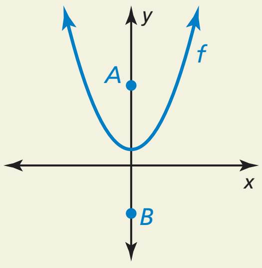
a. Write the function $T(x) = h(x) - g(x)$. What does $T(x)$ represent?

b. When your ball hits the ground, what is the height of your friend's ball? Use a graph to justify your answer.

34. **MAKING AN ARGUMENT** Your friend claims that in the equation $y = ax^2 + c$, the vertex changes when the value of a changes. Is your friend correct? Explain your reasoning.

35. **MATHEMATICAL CONNECTIONS** The area A (in square feet) of a square patio is represented by $A = x^2$, where x is the length of one side of the patio. You add 48 square feet to the patio, resulting in a total area of 192 square feet. What are the dimensions of the original patio? Use a graph to justify your answer.

36. **HOW DO YOU SEE IT?** The graph of $f(x) = ax^2 + c$ is shown. Points A and B are the same distance from the vertex of the graph of f. Which point is closer to the vertex of the graph of f as c increases?



37. **REASONING** Describe two methods you can use to find the zeros of the function $f(t) = -16t^2 + 400$. Check your answer by graphing.

38. **PROBLEM SOLVING** The paths of water from three different garden waterfalls are given below. Each function gives the height h (in feet) and the horizontal distance d (in feet) of the water.

Waterfall 1 $h = -3.1d^2 + 4.8$

Waterfall 2 $h = -3.5d^2 + 1.9$

Waterfall 3 $h = -1.1d^2 + 1.6$

a. Which waterfall drops water from the highest point?

b. Which waterfall follows the narrowest path?

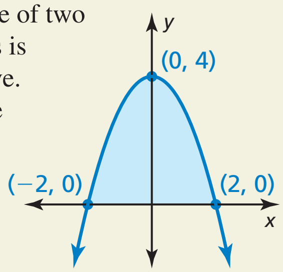
c. Which waterfall sends water the farthest?

39. **WRITING EQUATIONS** Two acorns fall to the ground from an oak tree. One falls 45 feet, while the other falls 32 feet.

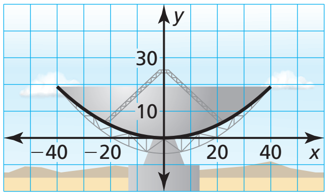
a. For each acorn, write an equation that represents the height h (in feet) as a function of the time t (in seconds).

b. Describe how the graphs of the two equations are related.

40. **THOUGHT PROVOKING** One of two classic problems in calculus is to find the area under a curve. Approximate the area of the region bounded by the parabola and the x-axis. Show your work.



41. **CRITICAL THINKING** A cross section of the parabolic surface of the antenna shown can be modeled by $y = 0.012x^2$, where x and y are measured in feet. The antenna is moved up so that the outer edges of the dish are 25 feet above the x-axis. Where is the vertex of the cross section located? Explain.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression when $a = 4$ and $b = -3$. *(Skills Review Handbook)*

42. $\dfrac{a}{4b}$

43. $-\dfrac{b}{2a}$

44. $\dfrac{a - b}{3a + b}$

45. $-\dfrac{b + 2a}{ab}$

8.3 Graphing $f(x) = ax^2 + bx + c$

Essential Question How can you find the vertex of the graph of $f(x) = ax^2 + bx + c$?

EXPLORATION 1 Comparing x-Intercepts with the Vertex

Work with a partner.

a. Sketch the graphs of $y = 2x^2 - 8x$ and $y = 2x^2 - 8x + 6$.

b. What do you notice about the x-coordinate of the vertex of each graph?

c. Use the graph of $y = 2x^2 - 8x$ to find its x-intercepts. Verify your answer by solving $0 = 2x^2 - 8x$.

d. Compare the value of the x-coordinate of the vertex with the values of the x-intercepts.

EXPLORATION 2 Finding x-Intercepts

Work with a partner.

a. Solve $0 = ax^2 + bx$ for x by factoring.

b. What are the x-intercepts of the graph of $y = ax^2 + bx$?

c. Copy and complete the table to verify your answer.

x	$y = ax^2 + bx$
0	
$-\frac{b}{a}$	

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to make conjectures and build a logical progression of statements.

EXPLORATION 3 Deductive Reasoning

Work with a partner. Complete the following logical argument.

The x-intercepts of the graph of $y = ax^2 + bx$ are 0 and $-\frac{b}{a}$.

The vertex of the graph of $y = ax^2 + bx$ occurs when $x =$ ▭.

The vertices of the graphs of $y = ax^2 + bx$ and $y = ax^2 + bx + c$ have the same x-coordinate.

The vertex of the graph of $y = ax^2 + bx + c$ occurs when $x =$ ▭.

Communicate Your Answer

4. How can you find the vertex of the graph of $f(x) = ax^2 + bx + c$?

5. Without graphing, find the vertex of the graph of $f(x) = x^2 - 4x + 3$. Check your result by graphing.

8.3 Lesson

What You Will Learn

- Graph quadratic functions of the form $f(x) = ax^2 + bx + c$.
- Find maximum and minimum values of quadratic functions.

Core Vocabulary

maximum value, *p. 433*
minimum value, *p. 433*

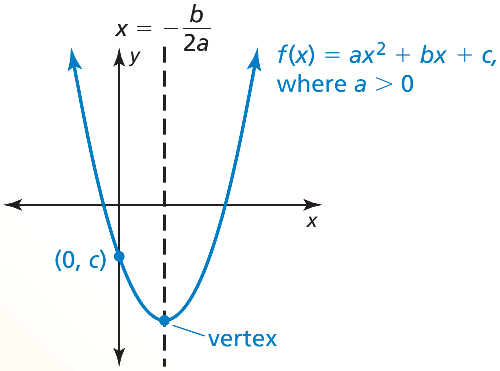
Previous
independent variable
dependent variable

Graphing $f(x) = ax^2 + bx + c$

Core Concept

Graphing $f(x) = ax^2 + bx + c$

- The graph opens up when $a > 0$, and the graph opens down when $a < 0$.
- The y-intercept is c.
- The x-coordinate of the vertex is $-\frac{b}{2a}$.
- The axis of symmetry is $x = -\frac{b}{2a}$.



EXAMPLE 1 **Finding the Axis of Symmetry and the Vertex**

Find (a) the axis of symmetry and (b) the vertex of the graph of $f(x) = 2x^2 + 8x - 1$.

SOLUTION

a. Find the axis of symmetry when $a = 2$ and $b = 8$.

$x = -\frac{b}{2a}$ Write the equation for the axis of symmetry.

$x = -\frac{8}{2(2)}$ Substitute 2 for *a* and 8 for *b*.

$x = -2$ Simplify.

The axis of symmetry is $x = -2$.

b. The axis of symmetry is $x = -2$, so the x-coordinate of the vertex is -2. Use the function to find the y-coordinate of the vertex.

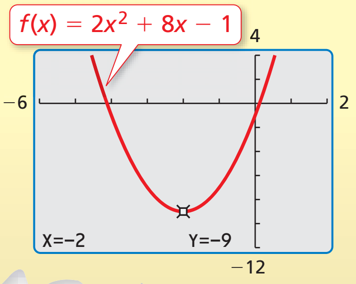
$f(x) = 2x^2 + 8x - 1$ Write the function.

$f(-2) = 2(-2)^2 + 8(-2) - 1$ Substitute -2 for *x*.

$= -9$ Simplify.

The vertex is $(-2, -9)$.

Check



Monitoring Progress

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Find (a) the axis of symmetry and (b) the vertex of the graph of the function.

1. $f(x) = 3x^2 - 2x$ **2.** $g(x) = x^2 + 6x + 5$ **3.** $h(x) = -\frac{1}{2}x^2 + 7x - 4$

EXAMPLE 2 Graphing $f(x) = ax^2 + bx + c$

Graph $f(x) = 3x^2 - 6x + 5$. Describe the domain and range.

COMMON ERROR

Be sure to include the negative sign before the fraction when finding the axis of symmetry.

SOLUTION

Step 1 Find and graph the axis of symmetry.

$$x = -\frac{b}{2a} = -\frac{(-6)}{2(3)} = 1 \qquad \text{Substitute and simplify.}$$

Step 2 Find and plot the vertex.

The axis of symmetry is $x = 1$, so the x-coordinate of the vertex is 1. Use the function to find the y-coordinate of the vertex.

$f(x) = 3x^2 - 6x + 5$ Write the function.

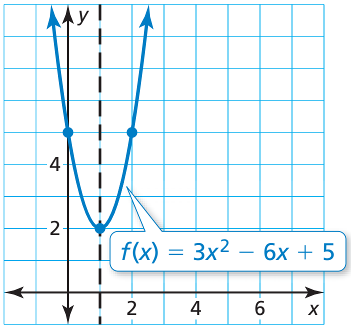
$f(1) = 3(1)^2 - 6(1) + 5$ Substitute 1 for x.

$= 2$ Simplify.

So, the vertex is (1, 2).

Step 3 Use the y-intercept to find two more points on the graph.

Because $c = 5$, the y-intercept is 5. So, (0, 5) lies on the graph. Because the axis of symmetry is $x = 1$, the point (2, 5) also lies on the graph.



Step 4 Draw a smooth curve through the points.

▶ The domain is all real numbers. The range is $y \geq 2$.

REMEMBER

The domain is the set of all possible input values of the independent variable x. The range is the set of all possible output values of the dependent variable y.

Monitoring Progress

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Graph the function. Describe the domain and range.

4. $h(x) = 2x^2 + 4x + 1$ **5.** $k(x) = x^2 - 8x + 7$ **6.** $p(x) = -5x^2 - 10x - 2$

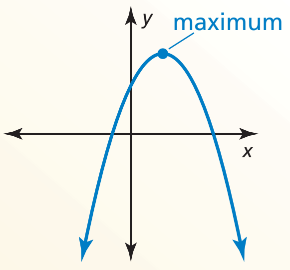
Finding Maximum and Minimum Values

Core Concept

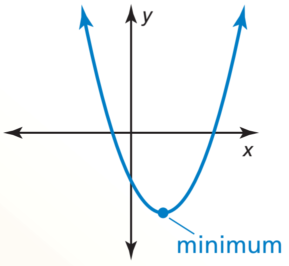
Maximum and Minimum Values

The y-coordinate of the vertex of the graph of $f(x) = ax^2 + bx + c$ is the **maximum value** of the function when $a < 0$ or the **minimum value** of the function when $a > 0$.

$f(x) = ax^2 + bx + c, a < 0$



$f(x) = ax^2 + bx + c, a > 0$



EXAMPLE 3 Finding a Maximum or Minimum Value

Tell whether the function $f(x) = -4x^2 - 24x - 19$ has a minimum value or a maximum value. Then find the value.

SOLUTION

For $f(x) = -4x^2 - 24x - 19$, $a = -4$ and $-4 < 0$. So, the parabola opens down and the function has a maximum value. To find the maximum value, find the y-coordinate of the vertex.

First, find the x-coordinate of the vertex. Use $a = -4$ and $b = -24$.

$$x = -\frac{b}{2a} = -\frac{(-24)}{2(-4)} = -3$$ Substitute and simplify.

Then evaluate the function when $x = -3$ to find the y-coordinate of the vertex.

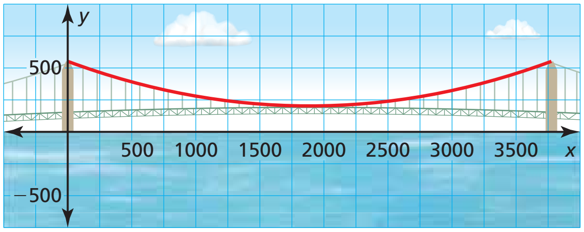
$$f(-3) = -4(-3)^2 - 24(-3) - 19$$ Substitute -3 for x.

$$= 17$$ Simplify.

▶ The maximum value is 17.

EXAMPLE 4 Finding a Minimum Value

The suspension cables between the two towers of the Mackinac Bridge in Michigan form a parabola that can be modeled by $y = 0.000098x^2 - 0.37x + 552$, where x and y are measured in feet. What is the height of the cable above the water at its lowest point?



SOLUTION

The lowest point of the cable is at the vertex of the parabola. Find the x-coordinate of the vertex. Use $a = 0.000098$ and $b = -0.37$.

$$x = -\frac{b}{2a} = -\frac{(-0.37)}{2(0.000098)} \approx 1888$$ Substitute and use a calculator.

Substitute 1888 for x in the equation to find the y-coordinate of the vertex.

$$y = 0.000098(1888)^2 - 0.37(1888) + 552 \approx 203$$

▶ The cable is about 203 feet above the water at its lowest point.

Monitoring Progress

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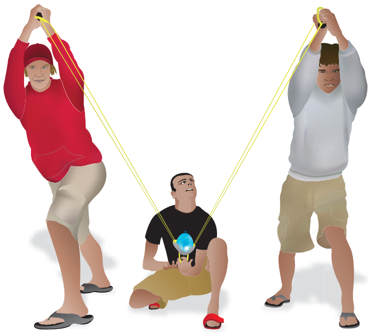
Tell whether the function has a minimum value or a maximum value. Then find the value.

7. $g(x) = 8x^2 - 8x + 6$

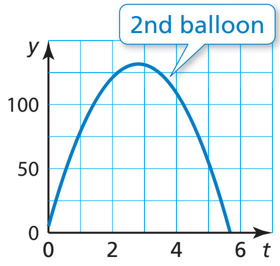
8. $h(x) = -\frac{1}{4}x^2 + 3x + 1$

9. The cables between the two towers of the Tacoma Narrows Bridge in Washington form a parabola that can be modeled by $y = 0.00016x^2 - 0.46x + 507$, where x and y are measured in feet. What is the height of the cable above the water at its lowest point?

EXAMPLE 5 Modeling with Mathematics



A group of friends is launching water balloons. The function $f(t) = -16t^2 + 80t + 5$ represents the height (in feet) of the first water balloon t seconds after it is launched. The height of the second water balloon t seconds after it is launched is shown in the graph. Which water balloon went higher?



SOLUTION

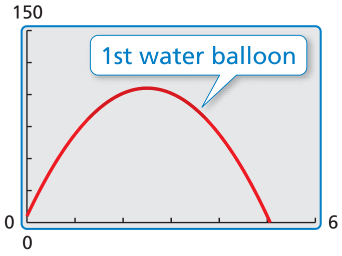
1. **Understand the Problem** You are given a function that represents the height of the first water balloon. The height of the second water balloon is represented graphically. You need to find and compare the maximum heights of the water balloons.

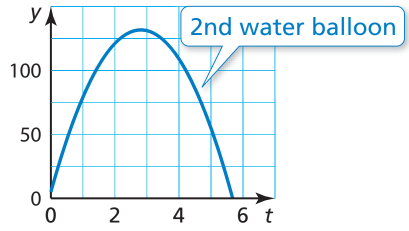
2. **Make a Plan** To compare the maximum heights, represent both functions graphically. Use a graphing calculator to graph $f(t) = -16t^2 + 80t + 5$ in an appropriate viewing window. Then visually compare the heights of the water balloons.

MODELING WITH MATHEMATICS

Because time cannot be negative, use only nonnegative values of t.

3. **Solve the Problem** Enter the function $f(t) = -16t^2 + 80t + 5$ into your calculator and graph it. Compare the graphs to determine which function has a greater maximum value.

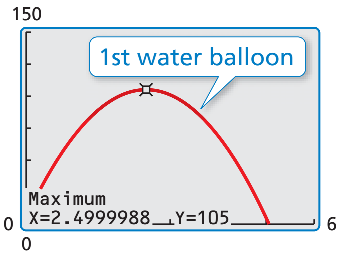




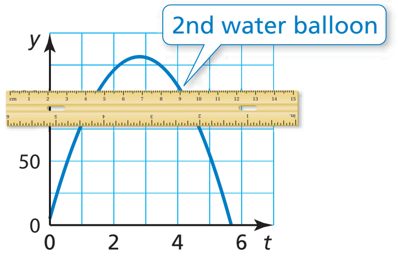
You can see that the second water balloon reaches a height of about 125 feet, while the first water balloon reaches a height of only about 100 feet.

▶ So, the second water balloon went higher.

4. **Look Back** Use the *maximum* feature to determine that the maximum value of $f(t) = -16t^2 + 80t + 5$ is 105. Use a straightedge to represent a height of 105 feet on the graph that represents the second water balloon to clearly see that the second water balloon went higher.



```
Maximum
X=2.4999988  Y=105
```



Monitoring Progress

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10. Which balloon is in the air longer? Explain your reasoning.

11. Which balloon reaches its maximum height faster? Explain your reasoning.

8.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Explain how you can tell whether a quadratic function has a maximum value or a minimum value without graphing the function.

2. **DIFFERENT WORDS, SAME QUESTION** Consider the quadratic function $f(x) = -2x^2 + 8x + 24$. Which is different? Find "both" answers.

What is the maximum value of the function?

What is the greatest number in the range of the function?

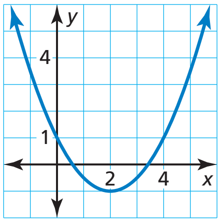
What is the y-coordinate of the vertex of the graph of the function?

What is the axis of symmetry of the graph of the function?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the vertex, the axis of symmetry, and the y-intercept of the graph.

3.



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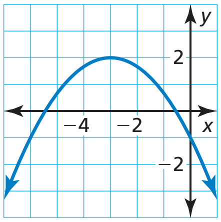

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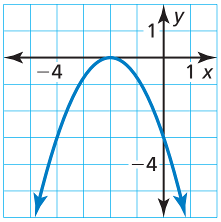

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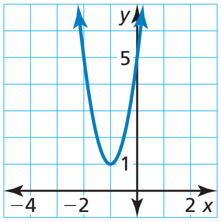


In Exercises 7–12, find (a) the axis of symmetry and (b) the vertex of the graph of the function. *(See Example 1.)*

7. $f(x) = 2x^2 - 4x$

8. $y = 3x^2 + 2x$

9. $y = -9x^2 - 18x - 1$

10. $f(x) = -6x^2 + 24x - 20$

11. $f(x) = \frac{2}{5}x^2 - 4x + 14$

12. $y = -\frac{3}{4}x^2 + 9x - 18$

In Exercises 13–18, graph the function. Describe the domain and range. *(See Example 2.)*

13. $f(x) = 2x^2 + 12x + 4$

14. $y = 4x^2 + 24x + 13$

15. $y = -8x^2 - 16x - 9$

16. $f(x) = -5x^2 + 20x - 7$

17. $y = \frac{2}{3}x^2 - 6x + 5$

18. $f(x) = -\frac{1}{2}x^2 - 3x - 4$

19. **ERROR ANALYSIS** Describe and correct the error in finding the axis of symmetry of the graph of $y = 3x^2 - 12x + 11$.

✗

$$x = -\frac{b}{2a} = \frac{-12}{2(3)} = -2$$

The axis of symmetry is $x = -2$.

20. **ERROR ANALYSIS** Describe and correct the error in graphing the function $f(x) = x^2 + 4x + 3$.

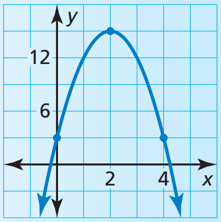
✗

The axis of symmetry is $x = \frac{b}{2a} = \frac{4}{2(1)} = 2$.

$f(2) = 2^2 + 4(2) + 3 = 15$

So, the vertex is $(2, 15)$.

The y-intercept is 3. So, the points $(0, 3)$ and $(4, 3)$ lie on the graph.



In Exercises 21–26, tell whether the function has a minimum value or a maximum value. Then find the value. *(See Example 3.)*

21. $y = 3x^2 - 18x + 15$

22. $f(x) = -5x^2 + 10x + 7$

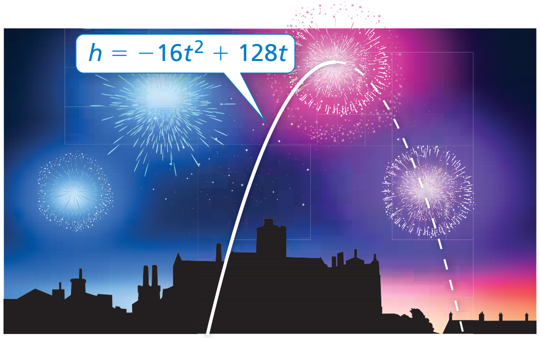
23. $f(x) = -4x^2 + 4x - 2$

24. $y = 2x^2 - 10x + 13$

25. $y = -\frac{1}{2}x^2 - 11x + 6$

26. $f(x) = \frac{1}{5}x^2 - 5x + 27$

27. **MODELING WITH MATHEMATICS** The function shown represents the height h (in feet) of a firework t seconds after it is launched. The firework explodes at its highest point. *(See Example 4.)*



a. When does the firework explode?

b. At what height does the firework explode?

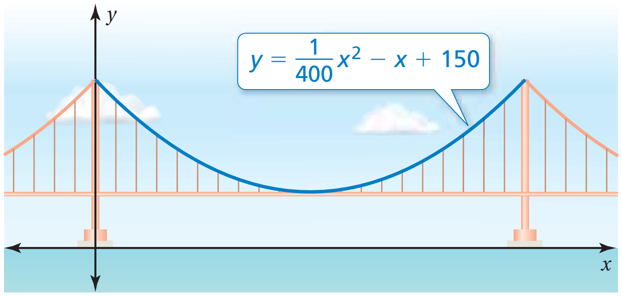
28. **MODELING WITH MATHEMATICS** The function $h(t) = -16t^2 + 16t$ represents the height (in feet) of a horse t seconds after it jumps during a steeplechase.

a. When does the horse reach its maximum height?

b. Can the horse clear a fence that is 3.5 feet tall? If so, by how much?

c. How long is the horse in the air?

29. **MODELING WITH MATHEMATICS** The cable between two towers of a suspension bridge can be modeled by the function shown, where x and y are measured in feet. The cable is at road level midway between the towers.



a. How far from each tower shown is the lowest point of the cable?

b. How high is the road above the water?

c. Describe the domain and range of the function shown.

30. **REASONING** Find the axis of symmetry of the graph of the equation $y = ax^2 + bx + c$ when $b = 0$. Can you find the axis of symmetry when $a = 0$? Explain.

31. **ATTENDING TO PRECISION** The vertex of a parabola is $(3, -1)$. One point on the parabola is $(6, 8)$. Find another point on the parabola. Justify your answer.

32. **MAKING AN ARGUMENT** Your friend claims that it is possible to draw a parabola through any two points with different x-coordinates. Is your friend correct? Explain.

USING TOOLS In Exercises 33–36, use the *minimum* or *maximum* feature of a graphing calculator to approximate the vertex of the graph of the function.

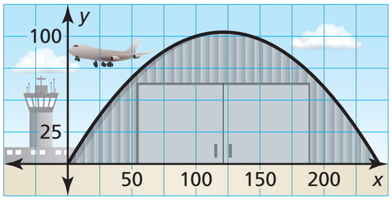
33. $y = 0.5x^2 + \sqrt{2}x - 3$

34. $y = -6.2x^2 + 4.8x - 1$

35. $y = -\pi x^2 + 3x$

36. $y = 0.25x^2 - 5^{2/3}x + 2$

37. **MODELING WITH MATHEMATICS** The opening of one aircraft hangar is a parabolic arch that can be modeled by the equation $y = -0.006x^2 + 1.5x$, where x and y are measured in feet. The opening of a second aircraft hangar is shown in the graph. *(See Example 5.)*



a. Which aircraft hangar is taller?

b. Which aircraft hangar is wider?

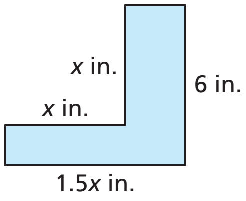
38. **MODELING WITH MATHEMATICS** An office supply store sells about 80 graphing calculators per month for \$120 each. For each \$6 decrease in price, the store expects to sell eight more calculators. The revenue from calculator sales is given by the function $R(n) = (\text{unit price})(\text{units sold})$, or $R(n) = (120 - 6n)(80 + 8n)$, where n is the number of \$6 price decreases.

a. How much should the store charge to maximize monthly revenue?

b. Using a different revenue model, the store expects to sell five more calculators for each \$4 decrease in price. Which revenue model results in a greater maximum monthly revenue? Explain.

MATHEMATICAL CONNECTIONS **In Exercises 39 and 40, (a) find the value of x that maximizes the area of the figure and (b) find the maximum area.**

39.



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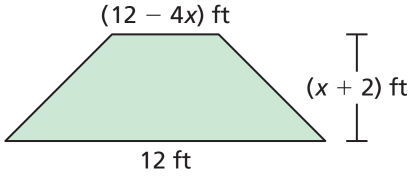
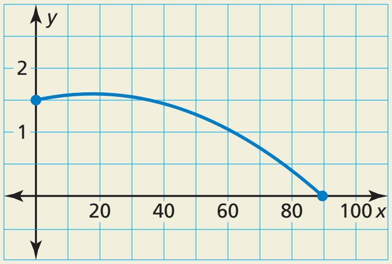


41. **WRITING** Compare the graph of $g(x) = x^2 + 4x + 1$ with the graph of $h(x) = x^2 - 4x + 1$.

42. **HOW DO YOU SEE IT?** During an archery competition, an archer shoots an arrow. The arrow follows the parabolic path shown, where x and y are measured in meters.



a. What is the initial height of the arrow?

b. Estimate the maximum height of the arrow.

c. How far does the arrow travel?

43. **USING TOOLS** The graph of a quadratic function passes through (3, 2), (4, 7), and (9, 2). Does the graph open up or down? Explain your reasoning.

44. **REASONING** For a quadratic function f, what does $f\left(-\frac{b}{2a}\right)$ represent? Explain your reasoning.

45. **PROBLEM SOLVING** Write a function of the form $y = ax^2 + bx$ whose graph contains the points (1, 6) and (3, 6).

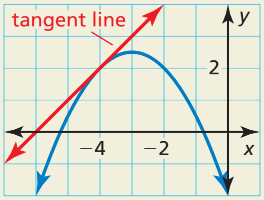
46. **CRITICAL THINKING** Parabolas A and B contain the points shown. Identify characteristics of each parabola, if possible. Explain your reasoning.

Parabola A	
x	y
2	3
6	4

Parabola B	
x	y
1	4
3	−4
5	4

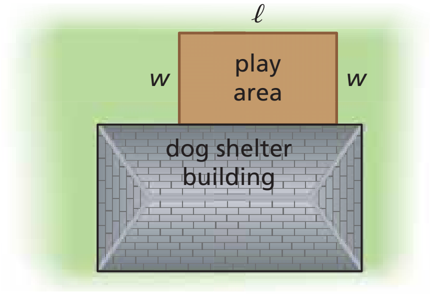
47. **MODELING WITH MATHEMATICS** At a basketball game, an air cannon launches T-shirts into the crowd. The function $y = -\frac{1}{8}x^2 + 4x$ represents the path of a T-shirt. The function $3y = 2x - 14$ represents the height of the bleachers. In both functions, y represents vertical height (in feet) and x represents horizontal distance (in feet). At what height does the T-shirt land in the bleachers?

48. **THOUGHT PROVOKING** One of two classic problems in calculus is finding the slope of a *tangent line* to a curve. An example of a tangent line, which just touches the parabola at one point, is shown.



Approximate the slope of the tangent line to the graph of $y = x^2$ at the point (1, 1). Explain your reasoning.

49. **PROBLEM SOLVING** The owners of a dog shelter want to enclose a rectangular play area on the side of their building. They have k feet of fencing. What is the maximum area of the outside enclosure in terms of k? (*Hint:* Find the y-coordinate of the vertex of the graph of the area function.)



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Describe the transformation(s) from the graph of $f(x) = |x|$ to the graph of the given function. *(Section 3.7)*

50. $q(x) = |x + 6|$

51. $h(x) = -0.5|x|$

52. $g(x) = |x - 2| + 5$

53. $p(x) = 3|x + 1|$

8.1–8.3 What Did You Learn?

Core Vocabulary

quadratic function, *p. 420*
parabola, *p. 420*
vertex, *p. 420*
axis of symmetry, *p. 420*
zero of a function, *p. 428*
maximum value, *p. 433*
minimum value, *p. 433*

Core Concepts

Section 8.1

Characteristics of Quadratic Functions, *p. 420*
Graphing $f(x) = ax^2$ When $a > 0$, *p. 421*
Graphing $f(x) = ax^2$ When $a < 0$, *p. 421*

Section 8.2

Graphing $f(x) = ax^2 + c$, *p. 426*

Section 8.3

Graphing $f(x) = ax^2 + bx + c$, *p. 432*
Maximum and Minimum Values, *p. 433*

Mathematical Practices

1. Explain your plan for solving Exercise 18 on page 423.
2. How does graphing a function in Exercise 27 on page 429 help you answer the questions?
3. What definition and characteristics of the graph of a quadratic function did you use to answer Exercise 44 on page 438?

Study Skills

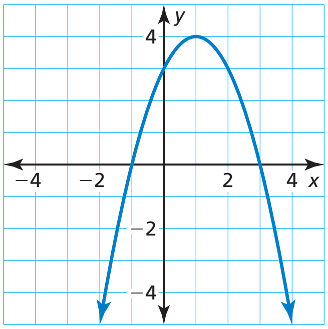
Learning Visually

- Draw a picture of a word problem before writing a verbal model. You do not have to be an artist.
- When making a review card for a word problem, include a picture. This will help you recall the information while taking a test.
- Make sure your notes are visually neat for easy recall.

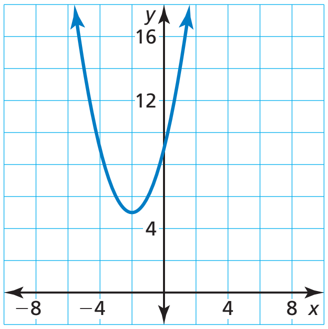
8.1–8.3 Quiz

Identify characteristics of the quadratic function and its graph. *(Section 8.1)*

1.



2.



Graph the function. Compare the graph to the graph of $f(x) = x^2$. *(Section 8.1 and Section 8.2)*

3. $h(x) = -x^2$

4. $p(x) = 2x^2 + 2$

5. $r(x) = 4x^2 - 16$

6. $b(x) = 8x^2$

7. $g(x) = \frac{2}{5}x^2$

8. $m(x) = -\frac{1}{2}x^2 - 4$

Describe the transformation from the graph of *f* to the graph of *g*. Then graph *f* and *g* in the same coordinate plane. Write an equation that represents *g* in terms of *x*. *(Section 8.2)*

9. $f(x) = 2x^2 + 1;\ g(x) = f(x) + 2$

10. $f(x) = -3x^2 + 12;\ g(x) = f(x) - 9$

11. $f(x) = \frac{1}{2}x^2 - 2;\ g(x) = f(x) - 6$

12. $f(x) = 5x^2 - 3;\ g(x) = f(x) + 1$

Graph the function. Describe the domain and range. *(Section 8.3)*

13. $f(x) = -4x^2 - 4x + 7$

14. $f(x) = 2x^2 + 12x + 5$

15. $y = x^2 + 4x - 5$

16. $y = -3x^2 + 6x + 9$

Tell whether the function has a minimum value or a maximum value. Then find the value. *(Section 8.3)*

17. $f(x) = 5x^2 + 10x - 3$

18. $f(x) = -\frac{1}{2}x^2 + 2x + 16$

19. $y = -x^2 + 4x + 12$

20. $y = 2x^2 + 8x + 3$

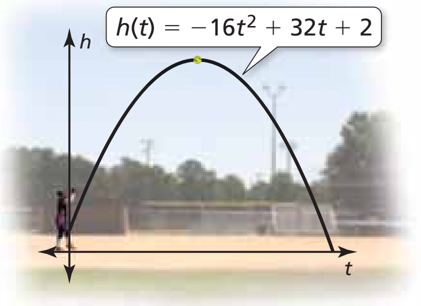
21. The distance y (in feet) that a coconut falls after t seconds is given by the function $y = 16t^2$. Use a graph to determine how many seconds it takes for the coconut to fall 64 feet. *(Section 8.1)*

22. The function $y = -16t^2 + 25$ represents the height y (in feet) of a pinecone t seconds after falling from a tree. *(Section 8.2)*

a. After how many seconds does the pinecone hit the ground?

b. A second pinecone falls from a height of 36 feet. Which pinecone hits the ground in the least amount of time? Explain.

23. The function shown models the height (in feet) of a softball t seconds after it is pitched in an underhand motion. Describe the domain and range. Find the maximum height of the softball. *(Section 8.3)*



8.4 Graphing $f(x) = a(x - h)^2 + k$

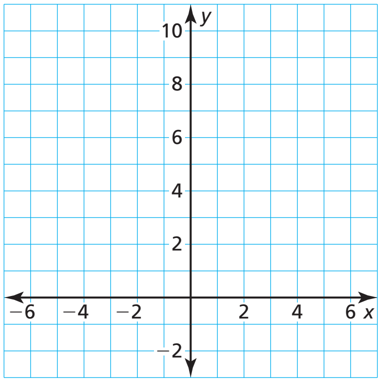
Essential Question How can you describe the graph of $f(x) = a(x - h)^2$?

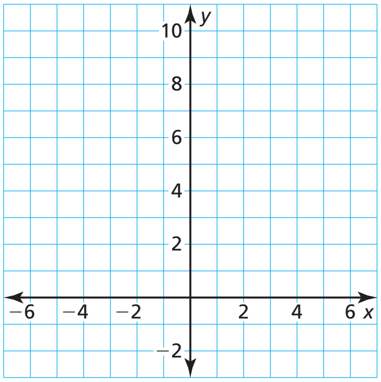
EXPLORATION 1 Graphing $y = a(x - h)^2$ When $h > 0$

Work with a partner. Sketch the graphs of the functions in the same coordinate plane. How does the value of h affect the graph of $y = a(x - h)^2$?

a. $f(x) = x^2$ and $g(x) = (x - 2)^2$

b. $f(x) = 2x^2$ and $g(x) = 2(x - 2)^2$



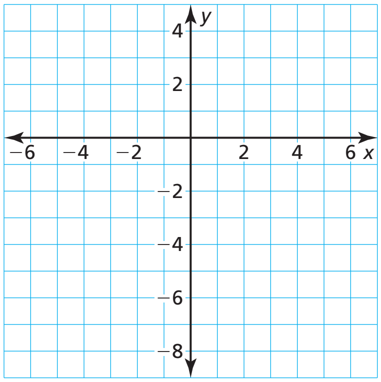


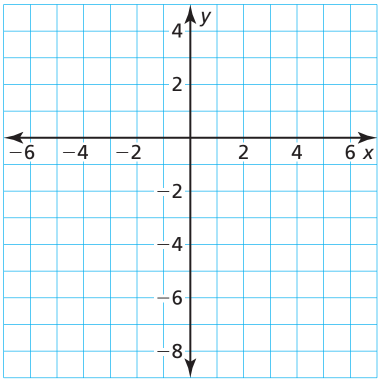
EXPLORATION 2 Graphing $y = a(x - h)^2$ When $h < 0$

Work with a partner. Sketch the graphs of the functions in the same coordinate plane. How does the value of h affect the graph of $y = a(x - h)^2$?

a. $f(x) = -x^2$ and $g(x) = -(x + 2)^2$

b. $f(x) = -2x^2$ and $g(x) = -2(x + 2)^2$





USING TOOLS STRATEGICALLY

To be proficient in math, you need to consider the available tools, such as a graphing calculator, when solving a mathematical problem.

Communicate Your Answer

3. How can you describe the graph of $f(x) = a(x - h)^2$?

4. Without graphing, describe the graph of each function. Use a graphing calculator to check your answer.

a. $y = (x - 3)^2$

b. $y = (x + 3)^2$

c. $y = -(x - 3)^2$

8.4 Lesson

Core Vocabulary

even function, *p. 442*
odd function, *p. 442*
vertex form (of a quadratic function), *p. 444*

Previous
reflection

What You Will Learn

- Identify even and odd functions.
- Graph quadratic functions of the form $f(x) = a(x - h)^2$.
- Graph quadratic functions of the form $f(x) = a(x - h)^2 + k$.
- Model real-life problems using $f(x) = a(x - h)^2 + k$.

Identifying Even and Odd Functions

Core Concept

Even and Odd Functions

A function $y = f(x)$ is **even** when $f(-x) = f(x)$ for each x in the domain of f. The graph of an even function is symmetric about the y-axis.

A function $y = f(x)$ is **odd** when $f(-x) = -f(x)$ for each x in the domain of f. The graph of an odd function is symmetric about the origin. A graph is *symmetric about the origin* when it looks the same after reflections in the x-axis and then in the y-axis.

STUDY TIP

The graph of an odd function looks the same after a 180° rotation about the origin.

EXAMPLE 1 Identifying Even and Odd Functions

Determine whether each function is *even*, *odd*, or *neither*.

a. $f(x) = 2x$ **b.** $g(x) = x^2 - 2$ **c.** $h(x) = 2x^2 + x - 2$

SOLUTION

a.
$f(x) = 2x$ Write the original function.
$f(-x) = 2(-x)$ Substitute $-x$ for x.
$= -2x$ Simplify.
$= -f(x)$ Substitute $f(x)$ for $2x$.

Because $f(-x) = -f(x)$, the function is odd.

b.
$g(x) = x^2 - 2$ Write the original function.
$g(-x) = (-x)^2 - 2$ Substitute $-x$ for x.
$= x^2 - 2$ Simplify.
$= g(x)$ Substitute $g(x)$ for $x^2 - 2$.

Because $g(-x) = g(x)$, the function is even.

c.
$h(x) = 2x^2 + x - 2$ Write the original function.
$h(-x) = 2(-x)^2 + (-x) - 2$ Substitute $-x$ for x.
$= 2x^2 - x - 2$ Simplify.

Because $h(x) = 2x^2 + x - 2$ and $-h(x) = -2x^2 - x + 2$, you can conclude that $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$. So, the function is neither even nor odd.

STUDY TIP

Most functions are neither even nor odd.

Monitoring Progress

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Determine whether the function is *even*, *odd*, or *neither*.

1. $f(x) = 5x$ **2.** $g(x) = 2^x$ **3.** $h(x) = 2x^2 + 3$

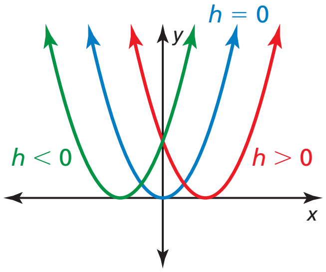
Graphing $f(x) = a(x - h)^2$

Core Concept

Graphing $f(x) = a(x - h)^2$

- When $h > 0$, the graph of $f(x) = a(x - h)^2$ is a horizontal translation h units right of the graph of $f(x) = ax^2$.
- When $h < 0$, the graph of $f(x) = a(x - h)^2$ is a horizontal translation $|h|$ units left of the graph of $f(x) = ax^2$.

The vertex of the graph of $f(x) = a(x - h)^2$ is $(h, 0)$, and the axis of symmetry is $x = h$.



EXAMPLE 2 Graphing $y = a(x - h)^2$

Graph $g(x) = \frac{1}{2}(x - 4)^2$. Compare the graph to the graph of $f(x) = x^2$.

SOLUTION

Step 1 Graph the axis of symmetry. Because $h = 4$, graph $x = 4$.

Step 2 Plot the vertex. Because $h = 4$, plot $(4, 0)$.

Step 3 Find and plot two more points on the graph. Choose two x-values less than the x-coordinate of the vertex. Then find $g(x)$ for each x-value.

When $x = 0$:	When $x = 2$:
$g(0) = \frac{1}{2}(0 - 4)^2$	$g(2) = \frac{1}{2}(2 - 4)^2$
$= 8$	$= 2$

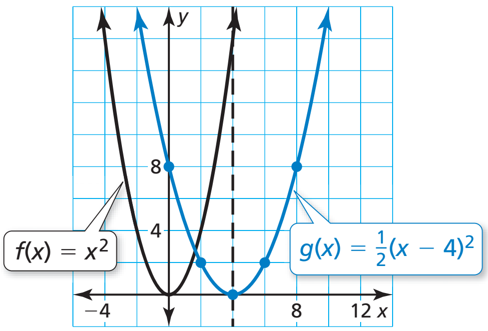
So, plot $(0, 8)$ and $(2, 2)$.

ANOTHER WAY

In Step 3, you could instead choose two x-values greater than the x-coordinate of the vertex.

Step 4 Reflect the points plotted in Step 3 in the axis of symmetry. So, plot $(8, 8)$ and $(6, 2)$.

Step 5 Draw a smooth curve through the points.



STUDY TIP

From the graph, you can see that $f(x) = x^2$ is an even function. However, $g(x) = \frac{1}{2}(x - 4)^2$ is neither even nor odd.

▶ Both graphs open up. The graph of g is wider than the graph of f. The axis of symmetry $x = 4$ and the vertex $(4, 0)$ of the graph of g are 4 units right of the axis of symmetry $x = 0$ and the vertex $(0, 0)$ of the graph of f. So, the graph of g is a translation 4 units right and a vertical shrink by a factor of $\frac{1}{2}$ of the graph of f.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

4. $g(x) = 2(x + 5)^2$

5. $h(x) = -(x - 2)^2$

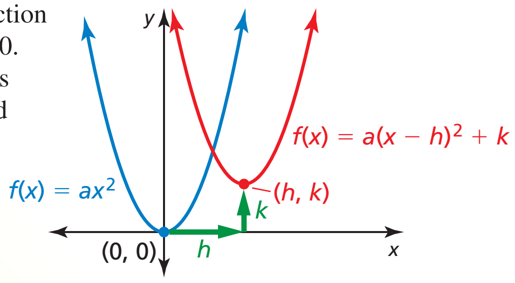
Graphing $f(x) = a(x - h)^2 + k$

Core Concept

Graphing $f(x) = a(x - h)^2 + k$

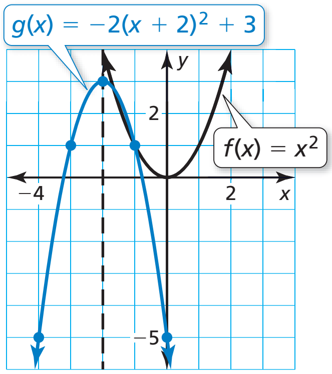
The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The graph of $f(x) = a(x - h)^2 + k$ is a translation h units horizontally and k units vertically of the graph of $f(x) = ax^2$.

The vertex of the graph of $f(x) = a(x - h)^2 + k$ is (h, k), and the axis of symmetry is $x = h$.



EXAMPLE 3 Graphing $y = a(x - h)^2 + k$

Graph $g(x) = -2(x + 2)^2 + 3$. Compare the graph to the graph of $f(x) = x^2$.



SOLUTION

Step 1 Graph the axis of symmetry. Because $h = -2$, graph $x = -2$.

Step 2 Plot the vertex. Because $h = -2$ and $k = 3$, plot $(-2, 3)$.

Step 3 Find and plot two more points on the graph. Choose two x-values less than the x-coordinate of the vertex. Then find $g(x)$ for each x-value. So, plot $(-4, -5)$ and $(-3, 1)$.

x	-4	-3
$g(x)$	-5	1

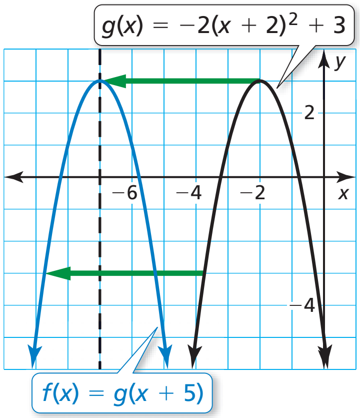
Step 4 Reflect the points plotted in Step 3 in the axis of symmetry. So, plot $(-1, 1)$ and $(0, -5)$.

Step 5 Draw a smooth curve through the points.

▶ The graph of g opens down and is narrower than the graph of f. The vertex of the graph of g, $(-2, 3)$, is 2 units left and 3 units up of the vertex of the graph of f, $(0, 0)$. So, the graph of g is a vertical stretch by a factor of 2, a reflection in the x-axis, and a translation 2 units left and 3 units up of the graph of f.

EXAMPLE 4 Transforming the Graph of $y = a(x - h)^2 + k$

Consider function g in Example 3. Graph $f(x) = g(x + 5)$.



SOLUTION

The function f is of the form $y = g(x - h)$, where $h = -5$. So, the graph of f is a horizontal translation 5 units left of the graph of g. To graph f, subtract 5 from the x-coordinates of the points on the graph of g.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

6. $g(x) = 3(x - 1)^2 + 6$

7. $h(x) = \frac{1}{2}(x + 4)^2 - 2$

8. Consider function g in Example 3. Graph $f(x) = g(x) - 3$.

Modeling Real-Life Problems

EXAMPLE 5 Modeling with Mathematics

Water fountains are usually designed to give a specific visual effect. For example, the water fountain shown consists of streams of water that are shaped like parabolas. Notice how the streams are designed to land on the underwater spotlights. Write and graph a quadratic function that models the path of a stream of water with a maximum height of 5 feet, represented by a vertex of (3, 5), landing on a spotlight 6 feet from the water jet, represented by (6, 0).

SOLUTION

1. **Understand the Problem** You know the vertex and another point on the graph that represents the parabolic path. You are asked to write and graph a quadratic function that models the path.

2. **Make a Plan** Use the given points and the vertex form to write a quadratic function. Then graph the function.

3. **Solve the Problem**
   Use the vertex form, vertex (3, 5), and point (6, 0) to find the value of a.

$f(x) = a(x - h)^2 + k$ Write the vertex form of a quadratic function.

$f(x) = a(x - 3)^2 + 5$ Substitute 3 for h and 5 for k.

$0 = a(6 - 3)^2 + 5$ Substitute 6 for x and 0 for $f(x)$.

$0 = 9a + 5$ Simplify.

$-\frac{5}{9} = a$ Solve for a.

So, $f(x) = -\frac{5}{9}(x - 3)^2 + 5$ models the path of a stream of water. Now graph the function.

Step 1 Graph the axis of symmetry. Because $h = 3$, graph $x = 3$.

Step 2 Plot the vertex, (3, 5).

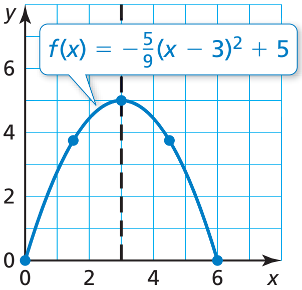
Step 3 Find and plot two more points on the graph. Because the x-axis represents the water surface, the graph should only contain points with nonnegative values of $f(x)$. You know that (6, 0) is on the graph. To find another point, choose an x-value between $x = 3$ and $x = 6$. Then find the corresponding value of $f(x)$.

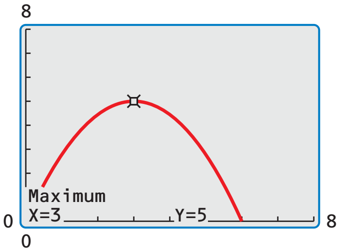
$f(4.5) = -\frac{5}{9}(4.5 - 3)^2 + 5 = 3.75$

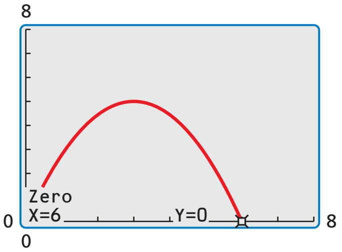
So, plot (6, 0) and (4.5, 3.75).

Step 4 Reflect the points plotted in Step 3 in the axis of symmetry. So, plot (0, 0) and (1.5, 3.75).

Step 5 Draw a smooth curve through the points.







4. **Look Back** Use a graphing calculator to graph $f(x) = -\frac{5}{9}(x - 3)^2 + 5$. Use the *maximum* feature to verify that the maximum value is 5. Then use the *zero* feature to verify that $x = 6$ is a zero of the function.

Monitoring Progress

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9. **WHAT IF?** The vertex is (3, 6). Write and graph a quadratic function that models the path.

8.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Compare the graph of an even function with the graph of an odd function.

2. **OPEN-ENDED** Write a quadratic function whose graph has a vertex of (1, 2).

3. **WRITING** Describe the transformation from the graph of $f(x) = ax^2$ to the graph of $g(x) = a(x - h)^2 + k$.

4. **WHICH ONE DOESN'T BELONG?** Which function does *not* belong with the other three? Explain your reasoning.

$f(x) = 8(x + 4)^2$ $f(x) = (x - 2)^2 + 4$ $f(x) = 2(x + 0)^2$ $f(x) = 3(x + 1)^2 + 1$

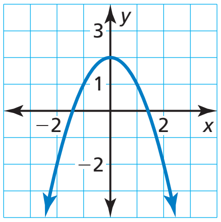
Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, determine whether the function is *even*, *odd*, or *neither*. *(See Example 1.)*

5. $f(x) = 4x + 3$
6. $g(x) = 3x^2$
7. $h(x) = 5^x + 2$
8. $m(x) = 2x^2 - 7x$
9. $p(x) = -x^2 + 8$
10. $f(x) = -\frac{1}{2}x$
11. $n(x) = 2x^2 - 7x + 3$
12. $r(x) = -6x^2 + 5$

In Exercises 13–18, determine whether the function represented by the graph is *even*, *odd*, or *neither*.

13.



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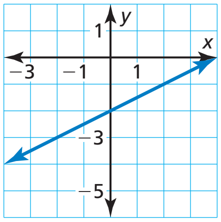

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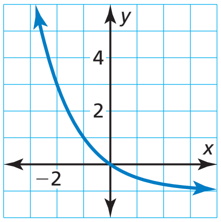

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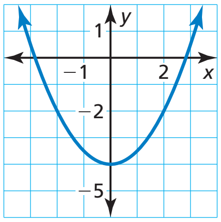

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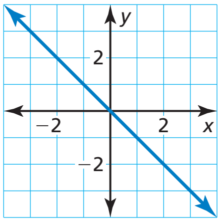

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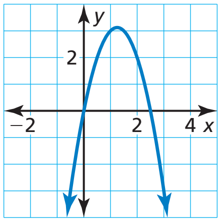


In Exercises 19–22, find the vertex and the axis of symmetry of the graph of the function.

19. $f(x) = 3(x + 1)^2$
20. $f(x) = \frac{1}{4}(x - 6)^2$
21. $y = -\frac{1}{8}(x - 4)^2$
22. $y = -5(x + 9)^2$

In Exercises 23–28, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 2.)*

23. $g(x) = 2(x + 3)^2$
24. $p(x) = 3(x - 1)^2$
25. $r(x) = \frac{1}{4}(x + 10)^2$
26. $n(x) = \frac{1}{3}(x - 6)^2$
27. $d(x) = \frac{1}{5}(x - 5)^2$
28. $q(x) = 6(x + 2)^2$

29. **ERROR ANALYSIS** Describe and correct the error in determining whether the function $f(x) = x^2 + 3$ is even, odd, or neither.

✗

$f(x) = x^2 + 3$
$f(-x) = (-x)^2 + 3$
$= x^2 + 3$
$= f(x)$

So, $f(x)$ is an odd function.

30. **ERROR ANALYSIS** Describe and correct the error in finding the vertex of the graph of the function.

✗

$y = -(x + 8)^2$
Because $h = -8$, the vertex is $(0, -8)$.

In Exercises 31–34, find the vertex and the axis of symmetry of the graph of the function.

31. $y = -6(x + 4)^2 - 3$

32. $f(x) = 3(x - 3)^2 + 6$

33. $f(x) = -4(x + 3)^2 + 1$

34. $y = -(x - 6)^2 - 5$

In Exercises 35–38, match the function with its graph.

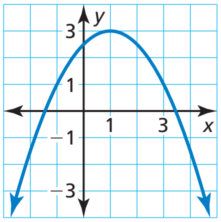
35. $y = -(x + 1)^2 - 3$

36. $y = -\frac{1}{2}(x - 1)^2 + 3$

37. $y = \frac{1}{3}(x - 1)^2 + 3$

38. $y = 2(x + 1)^2 - 3$

A.



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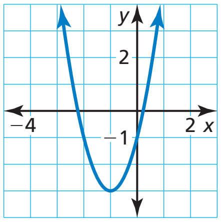
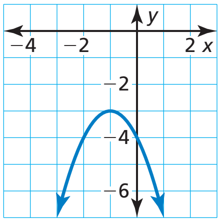


C.



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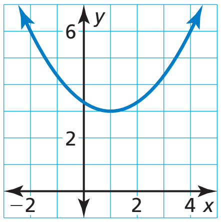


In Exercises 39–44, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 3.)*

39. $h(x) = (x - 2)^2 + 4$

40. $g(x) = (x + 1)^2 - 7$

41. $r(x) = 4(x - 1)^2 - 5$

42. $n(x) = -(x + 4)^2 + 2$

43. $g(x) = -\frac{1}{3}(x + 3)^2 - 2$

44. $r(x) = \frac{1}{2}(x - 2)^2 - 4$

In Exercises 45–48, let $f(x) = (x - 2)^2 + 1$. Match the function with its graph.

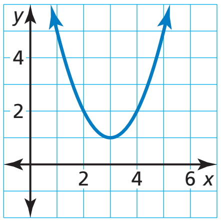
45. $g(x) = f(x - 1)$

46. $r(x) = f(x + 2)$

47. $h(x) = f(x) + 2$

48. $p(x) = f(x) - 3$

A.



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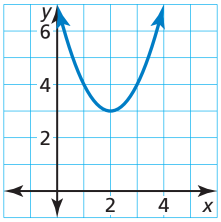

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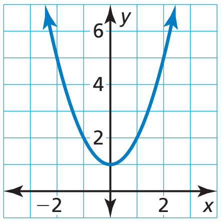

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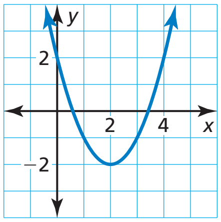


In Exercises 49–54, graph g. *(See Example 4.)*

49. $f(x) = 2(x - 1)^2 + 1$; $g(x) = f(x + 3)$

50. $f(x) = -(x + 1)^2 + 2$; $g(x) = \frac{1}{2}f(x)$

51. $f(x) = -3(x + 5)^2 - 6$; $g(x) = 2f(x)$

52. $f(x) = 5(x - 3)^2 - 1$; $g(x) = f(x) - 6$

53. $f(x) = (x + 3)^2 + 5$; $g(x) = f(x - 4)$

54. $f(x) = -2(x - 4)^2 - 8$; $g(x) = -f(x)$

55. **MODELING WITH MATHEMATICS** The height (in meters) of a bird diving to catch a fish is represented by $h(t) = 5(t - 2.5)^2$, where t is the number of seconds after beginning the dive.

a. Graph h.

b. Another bird's dive is represented by $r(t) = 2h(t)$. Graph r.

c. Compare the graphs. Which bird starts its dive from a greater height? Explain.

56. **MODELING WITH MATHEMATICS** A kicker punts a football. The height (in yards) of the football is represented by $f(x) = -\frac{1}{9}(x - 30)^2 + 25$, where x is the horizontal distance (in yards) from the kicker's goal line.

a. Graph f. Describe the domain and range.

b. On the next possession, the kicker punts the football. The height of the football is represented by $g(x) = f(x + 5)$. Graph g. Describe the domain and range.

c. Compare the graphs. On which possession does the kicker punt closer to his goal line? Explain.

In Exercises 57–62, write a quadratic function in vertex form whose graph has the given vertex and passes through the given point.

57. vertex: (1, 2); passes through (3, 10)

58. vertex: (−3, 5); passes through (0, −14)

59. vertex: (−2, −4); passes through (−1, −6)

60. vertex: (1, 8); passes through (3, 12)

61. vertex: (5, −2); passes through (7, 0)

62. vertex: (−5, −1); passes through (−2, 2)

63. **MODELING WITH MATHEMATICS** A portion of a roller coaster track is in the shape of a parabola. Write and graph a quadratic function that models this portion of the roller coaster with a maximum height of 90 feet, represented by a vertex of (25, 90), passing through the point (50, 0). *(See Example 5.)*

64. **MODELING WITH MATHEMATICS** A flare is launched from a boat and travels in a parabolic path until reaching the water. Write and graph a quadratic function that models the path of the flare with a maximum height of 300 meters, represented by a vertex of (59, 300), landing in the water at the point (119, 0).

In Exercises 65–68, rewrite the quadratic function in vertex form.

65. $y = 2x^2 - 8x + 4$

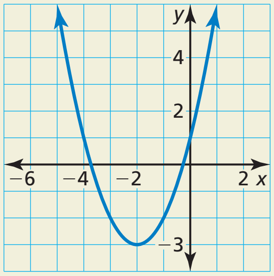
66. $y = 3x^2 + 6x - 1$

67. $f(x) = -5x^2 + 10x + 3$

68. $f(x) = -x^2 - 4x + 2$

69. **REASONING** Can a function be symmetric about the *x*-axis? Explain.

70. **HOW DO YOU SEE IT?** The graph of a quadratic function is shown. Determine which symbols to use to complete the vertex form of the quadratic function. Explain your reasoning.



$y = a(x\ \square\ 2)^2\ \square\ 3$

In Exercises 71–74, describe the transformation from the graph of *f* to the graph of *h*. Write an equation that represents *h* in terms of *x*.

71. $f(x) = -(x + 1)^2 - 2$
$h(x) = f(x) + 4$

72. $f(x) = 2(x - 1)^2 + 1$
$h(x) = f(x - 5)$

73. $f(x) = 4(x - 2)^2 + 3$
$h(x) = 2f(x)$

74. $f(x) = -(x + 5)^2 - 6$
$h(x) = \frac{1}{3}f(x)$

75. **REASONING** The graph of $y = x^2$ is translated 2 units right and 5 units down. Write an equation for the function in vertex form and in standard form. Describe advantages of writing the function in each form.

76. **THOUGHT PROVOKING** Which of the following are true? Justify your answers.

a. Any constant multiple of an even function is even.

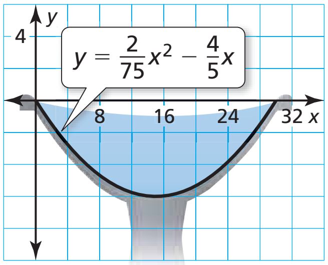
b. Any constant multiple of an odd function is odd.

c. The sum or difference of two even functions is even.

d. The sum or difference of two odd functions is odd.

e. The sum or difference of an even function and an odd function is odd.

77. **COMPARING FUNCTIONS** A cross section of a birdbath can be modeled by $y = \frac{1}{81}(x - 18)^2 - 4$, where *x* and *y* are measured in inches. The graph shows the cross section of another birdbath.



a. Which birdbath is deeper? Explain.

b. Which birdbath is wider? Explain.

78. **REASONING** Compare the graphs of $y = 2x^2 + 8x + 8$ and $y = x^2$ without graphing the functions. How can factoring help you compare the parabolas? Explain.

79. **MAKING AN ARGUMENT** Your friend says all absolute value functions are even because of their symmetry. Is your friend correct? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. *(Section 7.4)*

80. $x(x - 1) = 0$

81. $(x + 3)(x - 8) = 0$

82. $(3x - 9)(4x + 12) = 0$

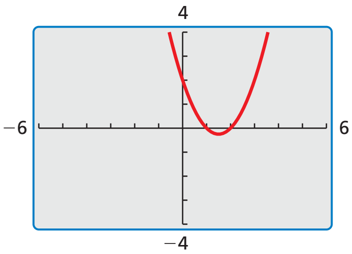
8.5 Using Intercept Form

Essential Question What are some of the characteristics of the graph of $f(x) = a(x - p)(x - q)$?

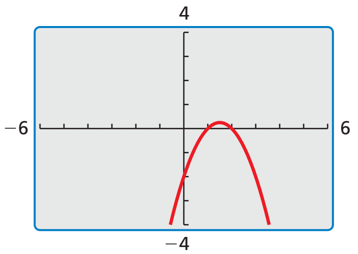
EXPLORATION 1 Using Zeros to Write Functions

Work with a partner. Each graph represents a function of the form $f(x) = (x - p)(x - q)$ or $f(x) = -(x - p)(x - q)$. Write the function represented by each graph. Explain your reasoning.

a.



b.



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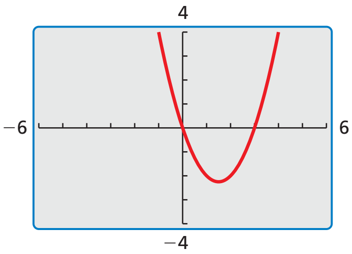

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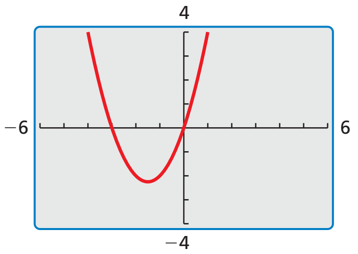
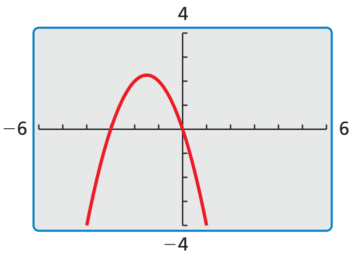


e.



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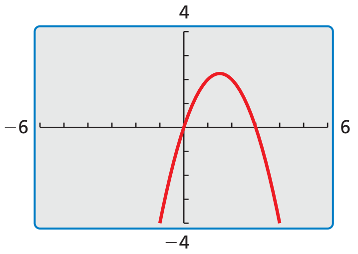

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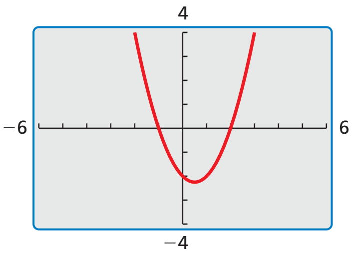

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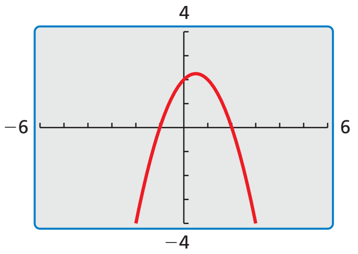


CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

Communicate Your Answer

2. What are some of the characteristics of the graph of $f(x) = a(x - p)(x - q)$?

3. Consider the graph of $f(x) = a(x - p)(x - q)$.

- **a.** Does changing the sign of a change the x-intercepts? Does changing the sign of a change the y-intercept? Explain your reasoning.
- **b.** Does changing the value of p change the x-intercepts? Does changing the value of p change the y-intercept? Explain your reasoning.

8.5 Lesson

Core Vocabulary

intercept form, *p. 450*

What You Will Learn

- Graph quadratic functions of the form $f(x) = a(x - p)(x - q)$.
- Use intercept form to find zeros of functions.
- Use characteristics to graph and write quadratic functions.
- Use characteristics to graph and write cubic functions.

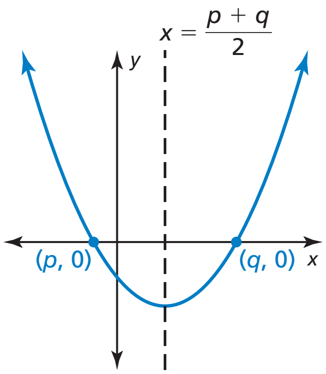
Graphing $f(x) = a(x - p)(x - q)$

You have already graphed quadratic functions written in several different forms, such as $f(x) = ax^2 + bx + c$ (standard form) and $g(x) = a(x - h)^2 + k$ (vertex form). Quadratic functions can also be written in **intercept form**, $f(x) = a(x - p)(x - q)$, where $a \neq 0$. In this form, the polynomial that defines a function is in factored form and the x-intercepts of the graph can be easily determined.

Core Concept

Graphing $f(x) = a(x - p)(x - q)$

- The x-intercepts are p and q.
- The axis of symmetry is halfway between $(p, 0)$ and $(q, 0)$. So, the axis of symmetry is $x = \frac{p + q}{2}$.
- The graph opens up when $a > 0$, and the graph opens down when $a < 0$.



EXAMPLE 1 Graphing $f(x) = a(x - p)(x - q)$

Graph $f(x) = -(x + 1)(x - 5)$. Describe the domain and range.

SOLUTION

Step 1 Identify the x-intercepts. Because the x-intercepts are $p = -1$ and $q = 5$, plot $(-1, 0)$ and $(5, 0)$.

Step 2 Find and graph the axis of symmetry.

$$x = \frac{p + q}{2} = \frac{-1 + 5}{2} = 2$$

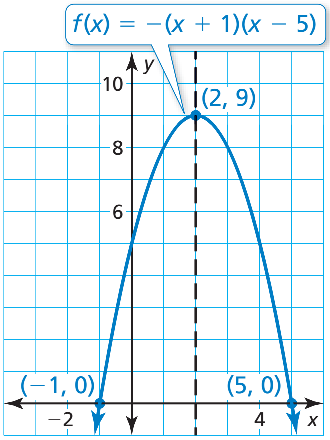
Step 3 Find and plot the vertex.

The x-coordinate of the vertex is 2. To find the y-coordinate of the vertex, substitute 2 for x and simplify.

$$f(2) = -(2 + 1)(2 - 5) = 9$$

So, the vertex is $(2, 9)$.

Step 4 Draw a parabola through the vertex and the points where the x-intercepts occur.



The domain is all real numbers. The range is $y \leq 9$.

EXAMPLE 2 Graphing a Quadratic Function

Graph $f(x) = 2x^2 - 8$. Describe the domain and range.

SOLUTION

Step 1 Rewrite the quadratic function in intercept form.

$f(x) = 2x^2 - 8$ Write the function.

$= 2(x^2 - 4)$ Factor out common factor.

$= 2(x + 2)(x - 2)$ Difference of two squares pattern

Step 2 Identify the x-intercepts. Because the x-intercepts are $p = -2$ and $q = 2$, plot $(-2, 0)$ and $(2, 0)$.

Step 3 Find and graph the axis of symmetry.

$$x = \frac{p + q}{2} = \frac{-2 + 2}{2} = 0$$

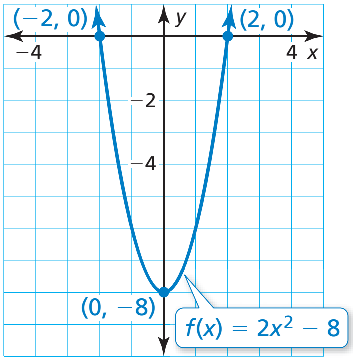
Step 4 Find and plot the vertex.

The x-coordinate of the vertex is 0.
The y-coordinate of the vertex is

$$f(0) = 2(0)^2 - 8 = -8.$$

So, the vertex is $(0, -8)$.

Step 5 Draw a parabola through the vertex and the points where the x-intercepts occur.



▶ The domain is all real numbers. The range is $y \geq -8$.

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Graph the quadratic function. Label the vertex, axis of symmetry, and x-intercepts. Describe the domain and range of the function.

1. $f(x) = (x + 2)(x - 3)$ **2.** $g(x) = -2(x - 4)(x + 1)$ **3.** $h(x) = 4x^2 - 36$

REMEMBER

Functions have zeros, and *graphs* have x-intercepts.

Using Intercept Form to Find Zeros of Functions

In Section 8.2, you learned that a zero of a function is an x-value for which $f(x) = 0$. You can use the intercept form of a function to find the zeros of the function.

EXAMPLE 3 Finding Zeros of a Function

Find the zeros of $f(x) = (x - 1)(x + 2)$.

SOLUTION

To find the zeros, determine the x-values for which $f(x)$ is 0.

$f(x) = (x - 1)(x + 2)$ Write the function.

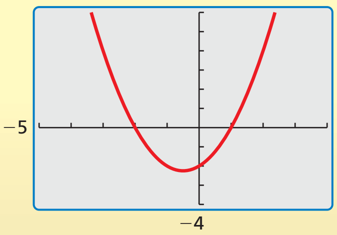
$0 = (x - 1)(x + 2)$ Substitute 0 for $f(x)$.

$x - 1 = 0$ *or* $x + 2 = 0$ Zero-Product Property

$x = 1$ *or* $x = -2$ Solve for x.

▶ So, the zeros of the function are −2 and 1.

Check



Core Concept

Factors and Zeros

For any factor $x - n$ of a polynomial, n is a zero of the function defined by the polynomial.

EXAMPLE 4 Finding Zeros of Functions

Find the zeros of each function.

a. $f(x) = -2x^2 - 10x - 12$

b. $h(x) = (x - 1)(x^2 - 16)$

SOLUTION

Write each function in intercept form to identify the zeros.

a. $f(x) = -2x^2 - 10x - 12$ Write the function.

$= -2(x^2 + 5x + 6)$ Factor out common factor.

$= -2(x + 3)(x + 2)$ Factor the trinomial.

▶ So, the zeros of the function are −3 and −2.

b. $h(x) = (x - 1)(x^2 - 16)$ Write the function.

$= (x - 1)(x + 4)(x - 4)$ Difference of two squares pattern

▶ So, the zeros of the function are −4, 1, and 4.

LOOKING FOR STRUCTURE

The function in Example 4(b) is called a *cubic function*. You can extend the concept of intercept form to cubic functions. You will graph a cubic function in Example 7.

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Find the zero(s) of the function.

4. $f(x) = (x - 6)(x - 1)$

5. $g(x) = 3x^2 - 12x + 12$

6. $h(x) = x(x^2 - 1)$

Using Characteristics to Graph and Write Quadratic Functions

EXAMPLE 5 Graphing a Quadratic Function Using Zeros

Use zeros to graph $h(x) = x^2 - 2x - 3$.

SOLUTION

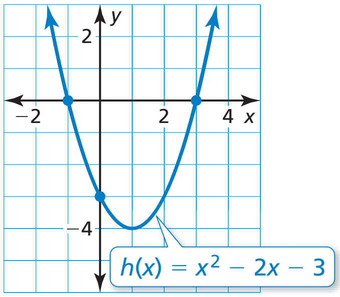
The function is in standard form. You know that the parabola opens up $(a > 0)$ and the y-intercept is −3. So, begin by plotting (0, −3).

Notice that the polynomial that defines the function is factorable. So, write the function in intercept form and identify the zeros.

$h(x) = x^2 - 2x - 3$ Write the function.

$= (x + 1)(x - 3)$ Factor the trinomial.

The zeros of the function are −1 and 3. So, plot (−1, 0) and (3, 0). Draw a parabola through the points.



ATTENDING TO PRECISION

To sketch a more precise graph, make a table of values and plot other points on the graph.

EXAMPLE 6 Writing Quadratic Functions

STUDY TIP

In part (a), many possible functions satisfy the given condition. The value a can be *any* nonzero number. To allow easier calculations, let $a = 1$. By letting $a = 2$, the resulting function would be $f(x) = 2x^2 + 12x + 22$.

Write a quadratic function in standard form whose graph satisfies the given condition(s).

a. vertex: $(-3, 4)$

b. passes through $(-9, 0)$, $(-2, 0)$, and $(-4, 20)$

SOLUTION

a. Because you know the vertex, use vertex form to write a function.

$f(x) = a(x - h)^2 + k$ Vertex form

$= 1(x + 3)^2 + 4$ Substitute for a, h, and k.

$= x^2 + 6x + 9 + 4$ Find the product $(x + 3)^2$.

$= x^2 + 6x + 13$ Combine like terms.

b. The given points indicate that the x-intercepts are -9 and -2. So, use intercept form to write a function.

$f(x) = a(x - p)(x - q)$ Intercept form

$= a(x + 9)(x + 2)$ Substitute for p and q.

Use the other given point, $(-4, 20)$, to find the value of a.

$20 = a(-4 + 9)(-4 + 2)$ Substitute -4 for x and 20 for $f(x)$.

$20 = a(5)(-2)$ Simplify.

$-2 = a$ Solve for a.

Use the value of a to write the function.

$f(x) = -2(x + 9)(x + 2)$ Substitute -2 for a.

$= -2x^2 - 22x - 36$ Simplify.

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Use zeros to graph the function.

7. $f(x) = (x - 1)(x - 4)$

8. $g(x) = x^2 + x - 12$

Write a quadratic function in standard form whose graph satisfies the given condition(s).

9. x-intercepts: -1 and 1

10. vertex: $(8, 8)$

11. passes through $(0, 0)$, $(10, 0)$, and $(4, 12)$

12. passes through $(-5, 0)$, $(4, 0)$, and $(3, -16)$

Using Characteristics to Graph and Write Cubic Functions

In Example 4, you extended the concept of intercept form to cubic functions.

$f(x) = a(x - p)(x - q)(x - r), a \neq 0$ Intercept form of a cubic function

The x-intercepts of the graph of f are p, q, and r.

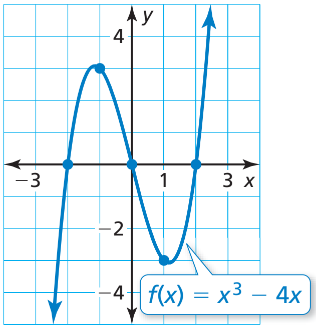
EXAMPLE 7 Graphing a Cubic Function Using Zeros

Use zeros to graph $f(x) = x^3 - 4x$.

SOLUTION

Notice that the polynomial that defines the function is factorable. So, write the function in intercept form and identify the zeros.

$$\begin{aligned} f(x) &= x^3 - 4x && \text{Write the function.} \\ &= x(x^2 - 4) && \text{Factor out } x. \\ &= x(x + 2)(x - 2) && \text{Difference of two squares pattern} \end{aligned}$$



The zeros of the function are −2, 0, and 2. So, plot (−2, 0), (0, 0), and (2, 0).

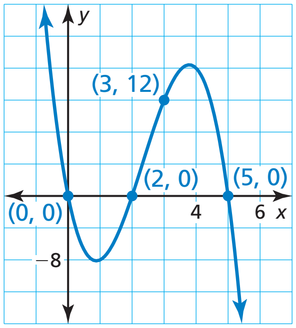
To help determine the shape of the graph, find points between the zeros.

x	−1	1
f(x)	3	−3

Plot (−1, 3) and (1, −3). Draw a smooth curve through the points.

EXAMPLE 8 Writing a Cubic Function

The graph represents a cubic function. Write the function.



SOLUTION

From the graph, you can see that the x-intercepts are 0, 2, and 5. Use intercept form to write a function.

$$\begin{aligned} f(x) &= a(x - p)(x - q)(x - r) && \text{Intercept form} \\ &= a(x - 0)(x - 2)(x - 5) && \text{Substitute for } p, q, \text{ and } r. \\ &= a(x)(x - 2)(x - 5) && \text{Simplify.} \end{aligned}$$

Use the other given point, (3, 12), to find the value of a.

$$\begin{aligned} 12 &= a(3)(3 - 2)(3 - 5) && \text{Substitute 3 for } x \text{ and 12 for } f(x). \\ -2 &= a && \text{Solve for } a. \end{aligned}$$

Use the value of a to write the function.

$$\begin{aligned} f(x) &= -2(x)(x - 2)(x - 5) && \text{Substitute } -2 \text{ for } a. \\ &= -2x^3 + 14x^2 - 20x && \text{Simplify.} \end{aligned}$$

▶ The function represented by the graph is $f(x) = -2x^3 + 14x^2 - 20x$.

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Use zeros to graph the function.

13. $g(x) = (x - 1)(x - 3)(x + 3)$

14. $h(x) = x^3 - 6x^2 + 5x$

15. The zeros of a cubic function are −3, −1, and 1. The graph of the function passes through the point (0, −3). Write the function.

8.5 Exercises

Vocabulary and Core Concept Check

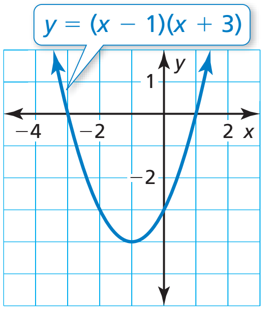
1. COMPLETE THE SENTENCE The values p and q are __________ of the graph of the function $f(x) = a(x - p)(x - q)$.

2. WRITING Explain how to find the maximum value or minimum value of a quadratic function when the function is given in intercept form.

Monitoring Progress and Modeling with Mathematics

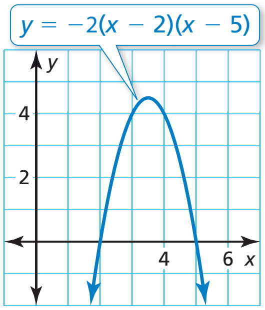
In Exercises 3–6, find the x-intercepts and axis of symmetry of the graph of the function.

3.



$y = (x - 1)(x + 3)$

4.



$y = -2(x - 2)(x - 5)$

5. $f(x) = -5(x + 7)(x - 5)$ **6.** $g(x) = \frac{2}{3}x(x + 8)$

In Exercises 7–12, graph the quadratic function. Label the vertex, axis of symmetry, and x-intercepts. Describe the domain and range of the function. *(See Example 1.)*

7. $f(x) = (x + 4)(x + 1)$ **8.** $y = (x - 2)(x + 2)$

9. $y = -(x + 6)(x - 4)$ **10.** $h(x) = -4(x - 7)(x - 3)$

11. $g(x) = 5(x + 1)(x + 2)$ **12.** $y = -2(x - 3)(x + 4)$

In Exercises 13–20, graph the quadratic function. Label the vertex, axis of symmetry, and x-intercepts. Describe the domain and range of the function. *(See Example 2.)*

13. $y = x^2 - 9$ **14.** $f(x) = x^2 - 8x$

15. $h(x) = -5x^2 + 5x$ **16.** $y = 3x^2 - 48$

17. $q(x) = x^2 + 9x + 14$ **18.** $p(x) = x^2 + 6x - 27$

19. $y = 4x^2 - 36x + 32$ **20.** $y = -2x^2 - 4x + 30$

In Exercises 21–30, find the zero(s) of the function. *(See Examples 3 and 4.)*

21. $y = -2(x - 2)(x - 10)$ **22.** $f(x) = \frac{1}{3}(x + 5)(x - 1)$

23. $g(x) = x^2 + 5x - 24$ **24.** $y = x^2 - 17x + 52$

25. $y = 3x^2 - 15x - 42$ **26.** $g(x) = -4x^2 - 8x - 4$

27. $f(x) = (x + 5)(x^2 - 4)$ **28.** $h(x) = (x^2 - 36)(x - 11)$

29. $y = x^3 - 49x$ **30.** $y = x^3 - x^2 - 9x + 9$

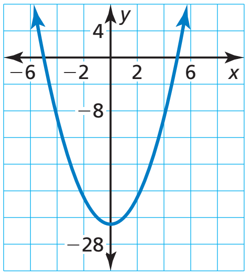
In Exercises 31–36, match the function with its graph.

31. $y = (x + 5)(x + 3)$ **32.** $y = (x + 5)(x - 3)$

33. $y = (x - 5)(x + 3)$ **34.** $y = (x - 5)(x - 3)$

35. $y = (x + 5)(x - 5)$ **36.** $y = (x + 3)(x - 3)$

A.



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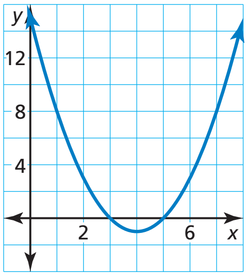

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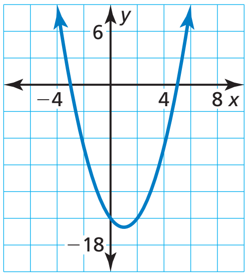

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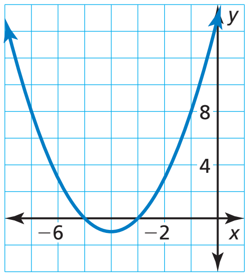

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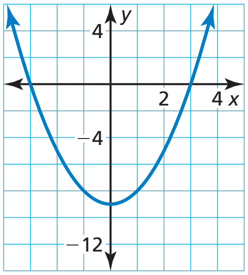

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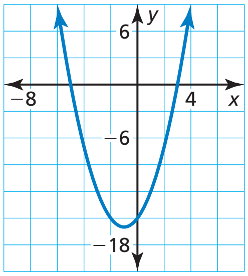


In Exercises 37–42, use zeros to graph the function. *(See Example 5.)*

37. $f(x) = (x + 2)(x - 6)$ **38.** $g(x) = -3(x + 1)(x + 7)$

39. $y = x^2 - 11x + 18$ **40.** $y = x^2 - x - 30$

41. $y = -5x^2 - 10x + 40$ **42.** $h(x) = 8x^2 - 8$

ERROR ANALYSIS **In Exercises 43 and 44, describe and correct the error in finding the zeros of the function.**

43.

✗ $y = 5(x + 3)(x - 2)$

The zeros of the function are 3 and −2.

44.

✗ $y = (x + 4)(x^2 - 9)$

The zeros of the function are −4 and 9.

In Exercises 45–56, write a quadratic function in standard form whose graph satisfies the given condition(s). *(See Example 6.)*

45. vertex: $(7, -3)$ **46.** vertex: $(4, 8)$

47. x-intercepts: 1 and 9 **48.** x-intercepts: −2 and −5

49. passes through $(-4, 0)$, $(3, 0)$, and $(2, -18)$

50. passes through $(-5, 0)$, $(-1, 0)$, and $(-4, 3)$

51. passes through $(7, 0)$

52. passes through $(0, 0)$ and $(6, 0)$

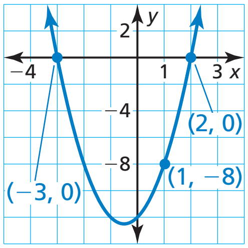
53. axis of symmetry: $x = -5$

54. y increases as x increases when $x < 4$; y decreases as x increases when $x > 4$.

55. range: $y \geq -3$ **56.** range: $y \leq 10$

In Exercises 57–60, write the quadratic function represented by the graph.

57.



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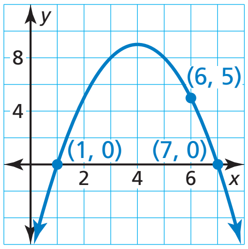

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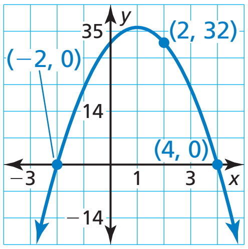

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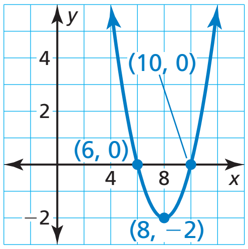


In Exercises 61–68, use zeros to graph the function. *(See Example 7.)*

61. $y = 5x(x + 2)(x - 6)$ **62.** $f(x) = -x(x + 9)(x + 3)$

63. $h(x) = (x - 2)(x + 2)(x + 7)$

64. $y = (x + 1)(x - 5)(x - 4)$

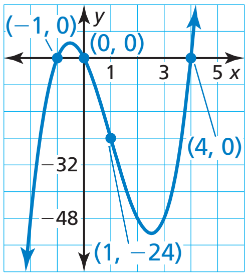
65. $f(x) = 3x^3 - 48x$ **66.** $y = -2x^3 + 20x^2 - 50x$

67. $y = -x^3 - 16x^2 - 28x$

68. $g(x) = 6x^3 + 30x^2 - 36x$

In Exercises 69–72, write the cubic function represented by the graph. *(See Example 8.)*

69.



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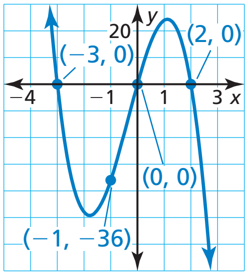

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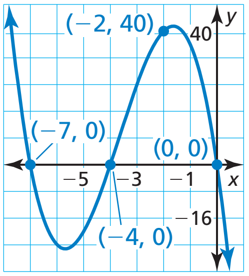
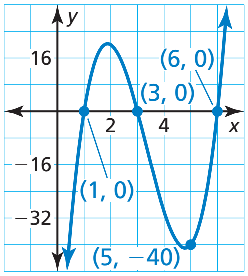


72.



In Exercises 73–76, write a cubic function whose graph satisfies the given condition(s).

73. x-intercepts: −2, 3, and 8

74. x-intercepts: −7, −5, and 0

75. passes through $(1, 0)$ and $(7, 0)$

76. passes through $(0, 6)$

In Exercises 77–80, all the zeros of a function are given. Use the zeros and the other point given to write a quadratic or cubic function represented by the table.

77.

x	y
0	0
2	30
7	0

78.

x	y
-3	0
1	-72
4	0

79.

x	y
-4	0
-3	0
0	-180
3	0

80.

x	y
-8	0
-6	-36
-3	0
0	0

In Exercises 81–84, sketch a parabola that satisfies the given conditions.

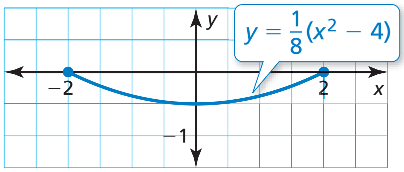
81. x-intercepts: -4 and 2; range: $y \geq -3$

82. axis of symmetry: $x = 6$; passes through (4, 15)

83. range: $y \leq 5$; passes through (0, 2)

84. x-intercept: 6; y-intercept: 1; range: $y \geq -4$

85. MODELING WITH MATHEMATICS Satellite dishes are shaped like parabolas to optimally receive signals. The cross section of a satellite dish can be modeled by the function shown, where x and y are measured in feet. The x-axis represents the top of the opening of the dish.

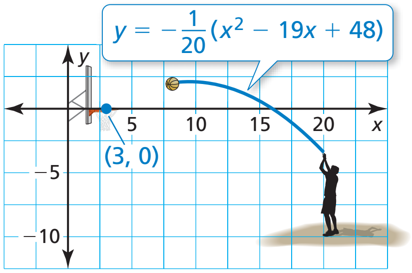


a. How wide is the satellite dish?

b. How deep is the satellite dish?

c. Write a quadratic function in standard form that models the cross section of a satellite dish that is 6 feet wide and 1.5 feet deep.

86. MODELING WITH MATHEMATICS A professional basketball player's shot is modeled by the function shown, where x and y are measured in feet.



a. Does the player make the shot? Explain.

b. The basketball player releases another shot from the point (13, 0) and makes the shot. The shot also passes through the point (10, 1.4). Write a quadratic function in standard form that models the path of the shot.

USING STRUCTURE **In Exercises 87–90, match the function with its graph.**

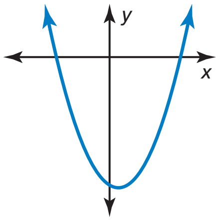
87. $y = -x^2 + 5x$

88. $y = x^2 - x - 12$

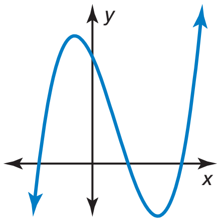
89. $y = x^3 - 2x^2 - 8x$

90. $y = x^3 - 4x^2 - 11x + 30$

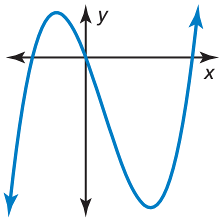
A.



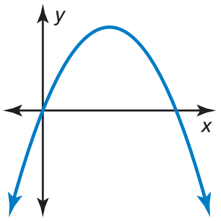
B.



C.



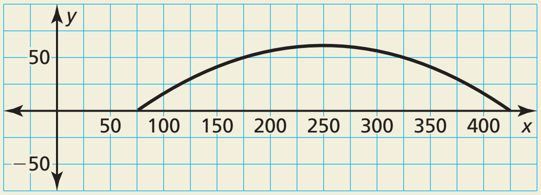
D.



91. CRITICAL THINKING Write a quadratic function represented by the table, if possible. If not, explain why.

x	-5	-3	-1	1
y	0	12	4	0

92. HOW DO YOU SEE IT? The graph shows the parabolic arch that supports the roof of a convention center, where x and y are measured in feet.



a. The arch can be represented by a function of the form $f(x) = a(x - p)(x - q)$. Estimate the values of p and q.

b. Estimate the width and height of the arch. Explain how you can use your height estimate to calculate a.

ANALYZING EQUATIONS In Exercises 93 and 94, (a) rewrite the quadratic function in intercept form and (b) graph the function using any method. Explain the method you used.

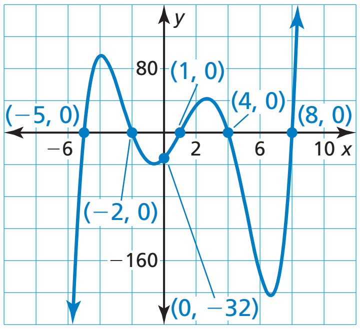
93. $f(x) = -3(x + 1)^2 + 27$

94. $g(x) = 2(x - 1)^2 - 2$

95. WRITING Can a quadratic function with exactly one real zero be written in intercept form? Explain.

96. MAKING AN ARGUMENT Your friend claims that any quadratic function can be written in standard form and in vertex form. Is your friend correct? Explain.

97. PROBLEM SOLVING Write the function represented by the graph in intercept form.



98. THOUGHT PROVOKING Sketch the graph of each function. Explain your procedure.

a. $f(x) = (x^2 - 1)(x^2 - 4)$

b. $g(x) = x(x^2 - 1)(x^2 - 4)$

99. REASONING Let k be a constant. Find the zeros of the function $f(x) = kx^2 - k^2x - 2k^3$ in terms of k.

PROBLEM SOLVING In Exercises 100 and 101, write a system of two quadratic equations whose graphs intersect at the given points. Explain your reasoning.

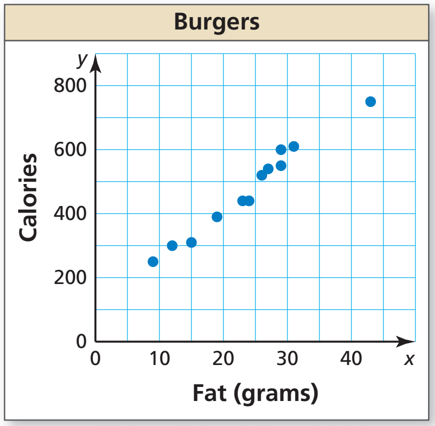
100. $(-4, 0)$ and $(2, 0)$

101. $(3, 6)$ and $(7, 6)$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

The scatter plot shows the amounts x (in grams) of fat and the numbers y of calories in 12 burgers at a fast-food restaurant. *(Section 4.4)*



102. How many calories are in the burger that contains 12 grams of fat?

103. How many grams of fat are in the burger that contains 600 calories?

104. What tends to happen to the number of calories as the number of grams of fat increases?

Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. *(Section 6.6)*

105. 3, 11, 21, 33, 47, . . .

106. −2, −6, −18, −54, . . .

107. 26, 18, 10, 2, −6, . . .

108. 4, 5, 9, 14, 23, . . .

8.6 Comparing Linear, Exponential, and Quadratic Functions

Essential Question How can you compare the growth rates of linear, exponential, and quadratic functions?

EXPLORATION 1 Comparing Speeds

Work with a partner. Three cars start traveling at the same time. The distance traveled in *t* minutes is *y* miles. Complete each table and sketch all three graphs in the same coordinate plane. Compare the speeds of the three cars. Which car has a constant speed? Which car is accelerating the most? Explain your reasoning.

t	$y = t$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

t	$y = 2^t - 1$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

t	$y = t^2$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

COMPARING PREDICTIONS

To be proficient in math, you need to visualize the results of varying assumptions, explore consequences, and compare predictions with data.

EXPLORATION 2 Comparing Speeds

Work with a partner. Analyze the speeds of the three cars over the given time periods. The distance traveled in *t* minutes is *y* miles. Which car eventually overtakes the others?

t	$y = t$
1.0	
1.5	
2.0	
2.5	
3.0	
3.5	
4.0	
4.5	
5.0	

t	$y = 2^t - 1$
1.0	
1.5	
2.0	
2.5	
3.0	
3.5	
4.0	
4.5	
5.0	

t	$y = t^2$
1.0	
1.5	
2.0	
2.5	
3.0	
3.5	
4.0	
4.5	
5.0	

Communicate Your Answer

3. How can you compare the growth rates of linear, exponential, and quadratic functions?

4. Which function has a growth rate that is eventually much greater than the growth rates of the other two functions? Explain your reasoning.

8.6 Lesson

Core Vocabulary

average rate of change, *p. 462*

Previous

slope

What You Will Learn

- Choose functions to model data.
- Write functions to model data.
- Compare functions using average rates of change.
- Solve real-life problems involving different function types.

Choosing Functions to Model Data

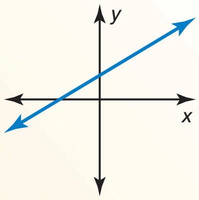
So far, you have studied linear functions, exponential functions, and quadratic functions. You can use these functions to model data.

Core Concept

Linear, Exponential, and Quadratic Functions

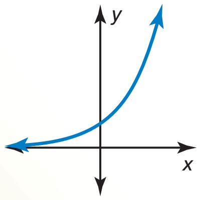
Linear Function

$y = mx + b$



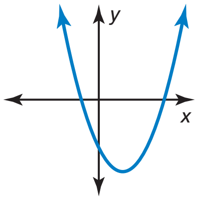
Exponential Function

$y = ab^x$



Quadratic Function

$y = ax^2 + bx + c$



EXAMPLE 1 **Using Graphs to Identify Functions**

Plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

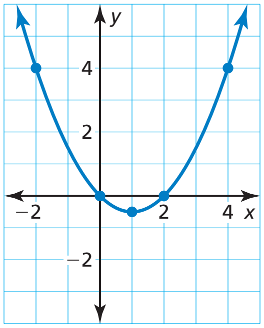
a. $(4, 4), (2, 0), (0, 0), \left(1, -\frac{1}{2}\right), (-2, 4)$

b. $(0, 1), (2, 4), (4, 7), (-2, -2), (-4, -5)$

c. $(0, 2), (2, 8), (1, 4), (-1, 1), \left(-2, \frac{1}{2}\right)$

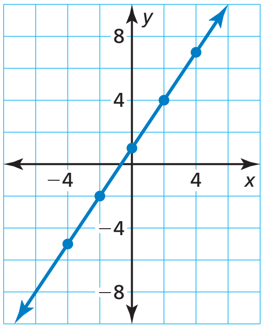
SOLUTION

a.



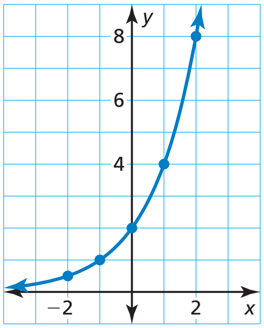
- quadratic

b.



- linear

c.



- exponential

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Plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

1. $(-1, 5), (2, -1), (0, -1), (3, 5), (1, -3)$

2. $(-1, 2), (-2, 8), (-3, 32), \left(0, \frac{1}{2}\right), \left(1, \frac{1}{8}\right)$

3. $(-3, 5), (0, -1), (2, -5), (-4, 7), (1, -3)$

Core Concept

Differences and Ratios of Functions

You can use patterns between consecutive data pairs to determine which type of function models the data. The differences of consecutive y-values are called *first differences*. The differences of consecutive first differences are called *second differences*.

- **Linear Function** The first differences are constant.
- **Exponential Function** Consecutive y-values have a common *ratio*.
- **Quadratic Function** The second differences are constant.

In all cases, the differences of consecutive x-values need to be constant.

STUDY TIP

The first differences for exponential and quadratic functions are *not* constant.

EXAMPLE 2 Using Differences or Ratios to Identify Functions

Tell whether each table of values represents a *linear*, an *exponential*, or a *quadratic* function.

a.

x	−3	−2	−1	0	1
y	11	8	5	2	−1

b.

x	−2	−1	0	1	2
y	1	2	4	8	16

c.

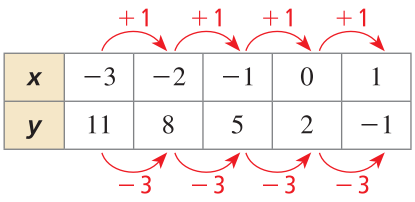
x	−2	−1	0	1	2
y	−1	−2	−1	2	7

STUDY TIP

First determine that the differences of consecutive x-values are constant. Then check whether the first differences are constant or consecutive y-values have a common ratio. If neither of these is true, check whether the second differences are constant.

SOLUTION

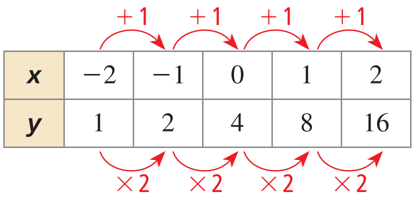
a.



x	−3	−2	−1	0	1
y	11	8	5	2	−1

▶ The first differences are constant. So, the table represents a linear function.

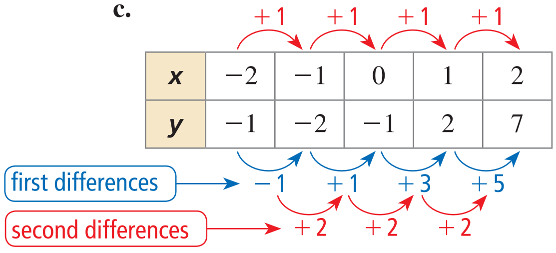
b.



x	−2	−1	0	1	2
y	1	2	4	8	16

▶ Consecutive y-values have a common ratio. So, the table represents an exponential function.

c.



x	−2	−1	0	1	2
y	−1	−2	−1	2	7

▶ The second differences are constant. So, the table represents a quadratic function.

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4. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function.

x	−1	0	1	2	3
y	1	3	9	27	81

Writing Functions to Model Data

EXAMPLE 3 Writing a Function to Model Data

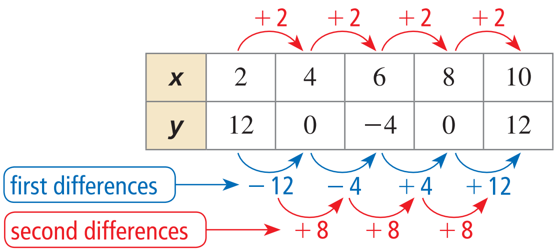
x	2	4	6	8	10
y	12	0	−4	0	12

Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

SOLUTION

Step 1 Determine which type of function the table of values represents.

The second differences are constant. So, the table represents a quadratic function.



x	2	4	6	8	10
y	12	0	−4	0	12

Step 2 Write an equation of the quadratic function. Using the table, notice that the x-intercepts are 4 and 8. So, use intercept form to write a function.

$y = a(x - 4)(x - 8)$ Substitute for p and q in intercept form.

Use another point from the table, such as (2, 12), to find a.

$12 = a(2 - 4)(2 - 8)$ Substitute 2 for x and 12 for y.

$1 = a$ Solve for a.

Use the value of a to write the function.

$y = (x - 4)(x - 8)$ Substitute 1 for a.

$= x^2 - 12x + 32$ Use the FOIL Method and combine like terms.

STUDY TIP

To check your function in Example 3, substitute the other points from the table to verify that they satisfy the function.

So, the quadratic function is $y = x^2 - 12x + 32$.

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5. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

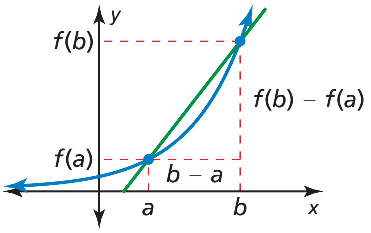
x	−1	0	1	2	3
y	16	8	4	2	1

Comparing Functions Using Average Rates of Change

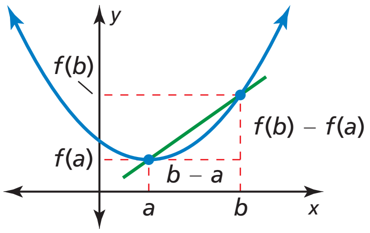
For nonlinear functions, the rate of change is not constant. You can compare two nonlinear functions over the same interval using their *average rates of change*. The **average rate of change** of a function $y = f(x)$ between $x = a$ and $x = b$ is the slope of the line through $(a, f(a))$ and $(b, f(b))$.

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

Exponential Function



Quadratic Function



Core Concept

Comparing Functions Using Average Rates of Change

- As a and b increase, the average rate of change between $x = a$ and $x = b$ of an increasing exponential function $y = f(x)$ will eventually exceed the average rate of change between $x = a$ and $x = b$ of an increasing quadratic function $y = g(x)$ or an increasing linear function $y = h(x)$. So, as x increases, $f(x)$ will eventually exceed $g(x)$ or $h(x)$.
- As a and b increase, the average rate of change between $x = a$ and $x = b$ of an increasing quadratic function $y = g(x)$ will eventually exceed the average rate of change between $x = a$ and $x = b$ of an increasing linear function $y = h(x)$. So, as x increases, $g(x)$ will eventually exceed $h(x)$.

STUDY TIP

You can explore these concepts using a graphing calculator.

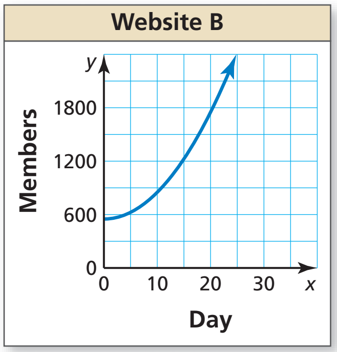
EXAMPLE 4 Using and Interpreting Average Rates of Change

Two social media websites open their memberships to the public. (a) Compare the websites by calculating and interpreting the average rates of change from Day 10 to Day 20. (b) Predict which website will have more members after 50 days. Explain.

Website A

Day, x	Members, y
0	650
5	1025
10	1400
15	1775
20	2150
25	2525

Website B



SOLUTION

a. Calculate the average rates of change by using the points whose x-coordinates are 10 and 20.

Website A: Use (10, 1400) and (20, 2150).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} = \frac{2150 - 1400}{20 - 10} = \frac{750}{10} = 75$$

Website B: Use the graph to estimate the points when $x = 10$ and $x = 20$. Use (10, 850) and (20, 1800).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} \approx \frac{1800 - 850}{20 - 10} = \frac{950}{10} = 95$$

▶ From Day 10 to Day 20, Website A membership increases at an average rate of 75 people per day, and Website B membership increases at an average rate of about 95 people per day. So, Website B membership is growing faster.

b. Using the table, membership increases and the average rates of change are constant. So, Website A membership can be represented by an increasing linear function. Using the graph, membership increases and the average rates of change are increasing. It appears that Website B membership can be represented by an increasing exponential or quadratic function.

After 25 days, the memberships of both websites are about equal and the average rate of change of Website B exceeds the average rate of change of Website A. So, Website B will have more members after 50 days.

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6. Compare the websites in Example 4 by calculating and interpreting the average rates of change from Day 0 to Day 10.

Solving Real-Life Problems

EXAMPLE 5 Comparing Different Function Types

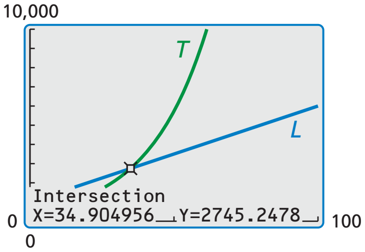
In 1900, Littleton had a population of 1000 people. Littleton's population increased by 50 people each year. In 1900, Tinyville had a population of 500 people. Tinyville's population increased by 5% each year.

a. In what year were the populations about equal?

b. Suppose Littleton's initial population doubled to 2000 and maintained a constant rate of increase of 50 people each year. Did Tinyville's population still catch up to Littleton's population? If so, in which year?

c. Suppose Littleton's rate of increase doubled to 100 people each year, in addition to doubling the initial population. Did Tinyville's population still catch up to Littleton's population? Explain.

SOLUTION



a. Let x represent the number of years since 1900. Write a function to model the population of each town.

Littleton: $L(x) = 50x + 1000$ Linear function

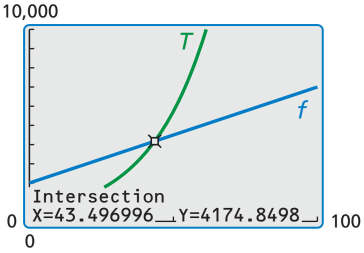
Tinyville: $T(x) = 500(1.05)^x$ Exponential function

Use a graphing calculator to graph each function in the same viewing window. Use the *intersect* feature to find the value of x for which $L(x) \approx T(x)$. The graphs intersect when $x \approx 34.9$.

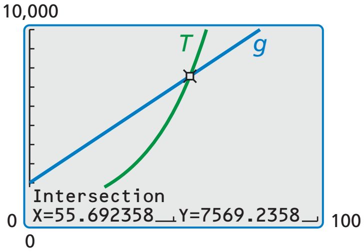
▶ So, the populations were about equal in 1934.

b. Littleton's new population function is $f(x) = 50x + 2000$. Use a graphing calculator to graph f and T in the same viewing window. Use the *intersect* feature to find the value of x for which $f(x) \approx T(x)$. The graphs intersect when $x \approx 43.5$.

▶ So, Tinyville's population caught Littleton's population in 1943.



c. Littleton's new population function is $g(x) = 100x + 2000$. Use a graphing calculator to graph g and T in the same viewing window. Use the *intersect* feature to find the value of x for which $g(x) \approx T(x)$. The graphs intersect when $x \approx 55.7$.



▶ So, Tinyville's population caught Littleton's population in 1955. Because Littleton's population shows linear growth and Tinyville's population shows exponential growth, Tinyville's population eventually exceeded Littleton's regardless of Littleton's constant rate or initial value.

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7. **WHAT IF?** Tinyville's population increased by 8% each year. In what year were the populations about equal?

8.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

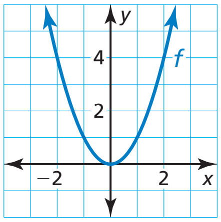
Vocabulary and Core Concept Check

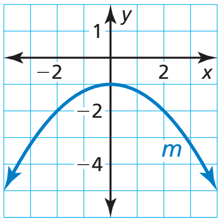
1. WRITING Name three types of functions that you can use to model data. Describe the equation and graph of each type of function.

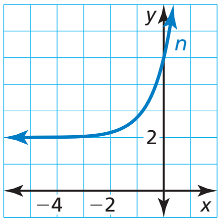
2. WRITING How can you decide whether to use a linear, an exponential, or a quadratic function to model a data set?

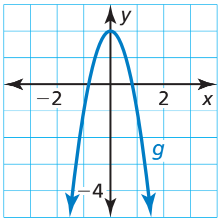
3. VOCABULARY Describe how to find the average rate of change of a function $y = f(x)$ between $x = a$ and $x = b$.

4. WHICH ONE DOESN'T BELONG? Which graph does *not* belong with the other three? Explain your reasoning.





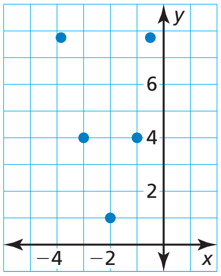




Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

5.



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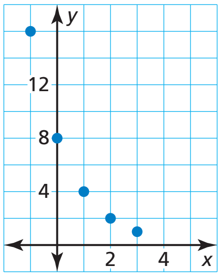

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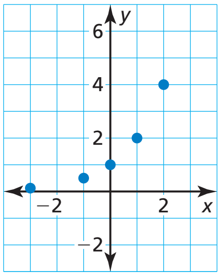

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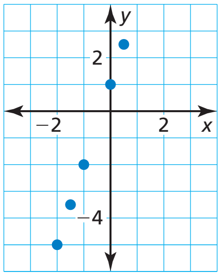


In Exercises 9–14, plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function. *(See Example 1.)*

9. (−2, −1), (−1, 0), (1, 2), (2, 3), (0, 1)

10. $\left(0, \frac{1}{4}\right)$, (1, 1), (2, 4), (3, 16), $\left(-1, \frac{1}{16}\right)$

11. (0, −3), (1, 0), (2, 9), (−2, 9), (−1, 0)

12. (−1, −3), (−3, 5), (0, −1), (1, 5), (2, 15)

13. (−4, −4), (−2, −3.4), (0, −3), (2, −2.6), (4, −2)

14. (0, 8), (−4, 0.25), (−3, 0.4), (−2, 1), (−1, 3)

In Exercises 15–18, tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. *(See Example 2.)*

15.

x	−2	−1	0	1	2
y	0	0.5	1	1.5	2

16.

x	−1	0	1	2	3
y	0.2	1	5	25	125

17.

x	2	3	4	5	6
y	2	6	18	54	162

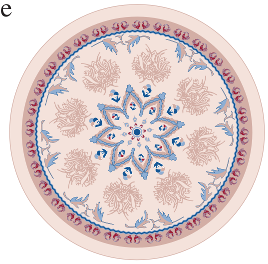
18.

x	−3	−2	−1	0	1
y	2	4.5	8	12.5	18

19. **MODELING WITH MATHEMATICS** A student takes a subway to a public library. The table shows the distances d (in miles) the student travels in t minutes. Let the time t represent the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain.

Time, t	0.5	1	3	5
Distance, d	0.335	0.67	2.01	3.35

20. **MODELING WITH MATHEMATICS** A store sells custom circular rugs. The table shows the costs c (in dollars) of rugs that have diameters of d feet. Let the diameter d represent the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain.



Diameter, d	3	4	5	6
Cost, c	63.90	113.60	177.50	255.60

In Exercises 21–26, tell whether the data represent a *linear*, an *exponential*, or a *quadratic* function. Then write the function. *(See Example 3.)*

21. $(-2, 8), (-1, 0), (0, -4), (1, -4), (2, 0), (3, 8)$

22. $(-3, 8), (-2, 4), (-1, 2), (0, 1), (1, 0.5)$

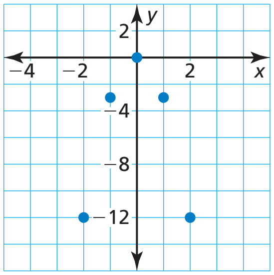
23.

x	-2	-1	0	1	2
y	4	1	-2	-5	-8

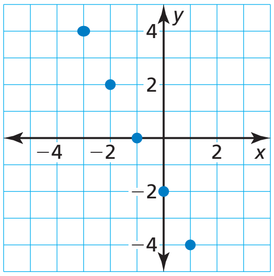
24.

x	-1	0	1	2	3
y	2.5	5	10	20	40

25.



26.



27. **ERROR ANALYSIS** Describe and correct the error in determining whether the table represents a linear, an exponential, or a quadratic function.

+1 +1 +1 +1

x	1	2	3	4	5
y	3	9	27	81	243

×3 ×3 ×3 ×3

Consecutive y-values change by a constant amount. So, the table represents a linear function.

28. **ERROR ANALYSIS** Describe and correct the error in writing the function represented by the table.

+1 +1 +1 +1

x	-3	-2	-1	0	1
y	4	0	-2	-2	0

first differences → -4 -2 $+0$ $+2$

second differences → $+2$ $+2$ $+2$

The table represents a quadratic function.

$$f(x) = a(x - 2)(x - 1)$$
$$4 = a(-3 - 2)(-3 - 1)$$
$$\frac{1}{5} = a$$
$$f(x) = \frac{1}{5}(x - 2)(x - 1)$$
$$= \frac{1}{5}x^2 - \frac{3}{5}x + \frac{2}{5}$$

So, the function is $f(x) = \frac{1}{5}x^2 - \frac{3}{5}x + \frac{2}{5}$.

29. **REASONING** The table shows the numbers of people attending the first five football games at a high school.

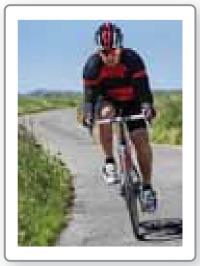
Game, g	1	2	3	4	5
People, p	252	325	270	249	310

a. Plot the points. Let the game g represent the independent variable.

b. Can a linear, an exponential, or a quadratic function represent this situation? Explain.

30. **MODELING WITH MATHEMATICS** The table shows the breathing rates y (in liters of air per minute) of a cyclist traveling at different speeds x (in miles per hour).

Speed, x	20	21	22	23	24
Breathing rate, y	51.4	57.1	63.3	70.3	78.0



a. Plot the points. Let the speed x represent the independent variable. Then determine the type of function that best represents this situation.

b. Write a function that models the data.

c. Find the breathing rate of a cyclist traveling 18 miles per hour. Round your answer to the nearest tenth.

31. **ANALYZING RATES OF CHANGE** The function $f(t) = -16t^2 + 48t + 3$ represents the height (in feet) of a volleyball t seconds after it is hit into the air.

a. Copy and complete the table.

t	0	0.5	1	1.5	2	2.5	3
$f(t)$							

b. Plot the ordered pairs and draw a smooth curve through the points.

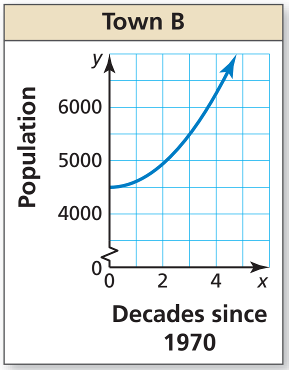
c. Describe where the function is increasing and decreasing.

d. Find the average rate of change for each 0.5-second interval in the table. What do you notice about the average rates of change when the function is increasing? decreasing?

32. **ANALYZING RELATIONSHIPS** The population of Town A in 1970 was 3000. The population of Town A increased by 20% every decade. Let x represent the number of decades since 1970. The graph shows the population of Town B. *(See Example 4.)*

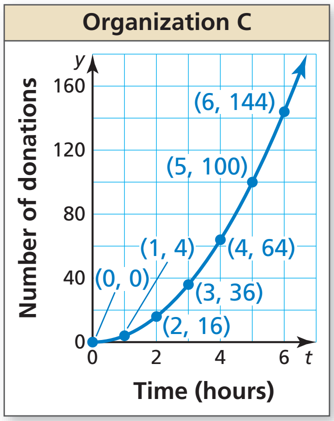
a. Compare the populations of the towns by calculating and interpreting the average rates of change from 1990 to 2010.

b. Predict which town will have a greater population after 2030. Explain.



33. **ANALYZING RELATIONSHIPS** Three organizations are collecting donations for a cause. Organization A begins with one donation, and the number of donations quadruples each hour. The table shows the numbers of donations collected by Organization B. The graph shows the numbers of donations collected by Organization C.

Time (hours), t	**Number of donations, y**
0	0
1	4
2	8
3	12
4	16
5	20
6	24



a. What type of function represents the numbers of donations collected by Organization A? B? C?

b. Find the average rates of change of each function for each 1-hour interval from $t = 0$ to $t = 6$.

c. For which function does the average rate of change increase most quickly? What does this tell you about the numbers of donations collected by the three organizations?

34. **COMPARING FUNCTIONS** The room expenses for two different resorts are shown. *(See Example 5.)*

Blue Water Resort

3 Night Stay $1100^{00}

Add additional nights to your stay for $112/night

Sea Breeze Resort

3 Night Stay $1000

4 Night Stay $1100

5 Night Stay $1210

Vacations available up to 14 nights. Each additional night is a 10% increase in price from the previous package.

a. For what length of vacation does each resort cost about the same?

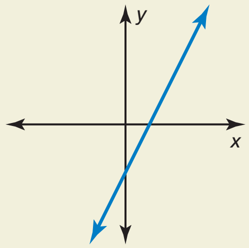
b. Suppose Blue Water Resort charges $1450 for the first three nights and $105 for each additional night. Would Sea Breeze Resort ever be more expensive than Blue Water Resort? Explain.

c. Suppose Sea Breeze Resort charges $1200 for the first three nights. The charge increases 10% for each additional night. Would Blue Water Resort ever be more expensive than Sea Breeze Resort? Explain.

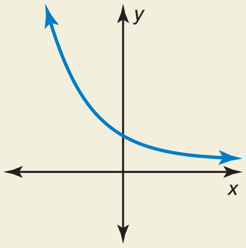
35. REASONING Explain why the average rate of change of a linear function is constant and the average rate of change of a quadratic or exponential function is not constant.

36. HOW DO YOU SEE IT? Match each graph with its function. Explain your reasoning.

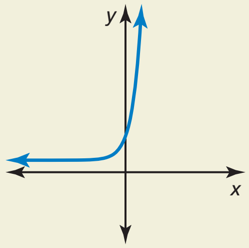
a.



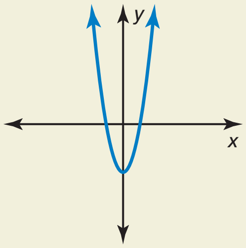
b.



c.



d.



A. $y = 2x^2 - 4$

B. $y = 2(4)^x + 1$

C. $y = 2\left(\frac{3}{4}\right)^x + 1$

D. $y = 2x - 4$

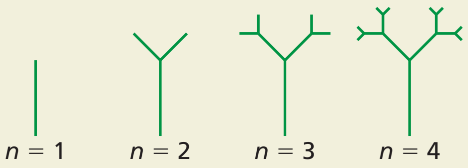
37. CRITICAL THINKING In the ordered pairs below, the y-values are given in terms of n. Tell whether the ordered pairs represent a *linear*, an *exponential*, or a *quadratic* function. Explain.

$(1, 3n - 1), (2, 10n + 2), (3, 26n),$
$(4, 51n - 7), (5, 85n - 19)$

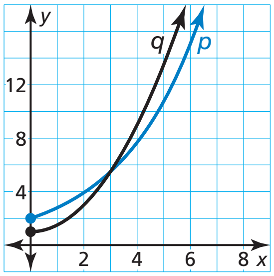
38. USING STRUCTURE Write a function that has constant second differences of 3.

39. CRITICAL THINKING Is the graph of a set of points enough to determine whether the points represent a linear, an exponential, or a quadratic function? Justify your answer.

40. THOUGHT PROVOKING Find four different patterns in the figure. Determine whether each pattern represents a *linear*, an *exponential*, or a *quadratic* function. Write a model for each pattern.



41. MAKING AN ARGUMENT Function p is an exponential function and function q is a quadratic function. Your friend says that after about $x = 3$, function q will always have a greater y-value than function p. Is your friend correct? Explain.



42. USING TOOLS The table shows the amount a (in billions of dollars) United States residents spent on pets or pet-related products and services each year for a 5-year period. Let the year x represent the independent variable. Using technology, find a function that models the data. How did you choose the model? Predict how much residents will spend on pets or pet-related products and services in Year 7.

Year, x	1	2	3	4	5
Amount, a	53.1	56.9	61.8	65.7	67.1

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression. *(Section 6.2)*

43. $\sqrt{121}$

44. $\sqrt[3]{125}$

45. $\sqrt[3]{512}$

46. $\sqrt[5]{243}$

Find the product. *(Section 7.3)*

47. $(x + 8)(x - 8)$

48. $(4y + 2)(4y - 2)$

49. $(3a - 5b)(3a + 5b)$

50. $(-2r + 6s)(-2r - 6s)$

8.4–8.6 What Did You Learn?

Core Vocabulary

even function, *p. 442*
odd function, *p. 442*
vertex form (of a quadratic function), *p. 444*
intercept form, *p. 450*
average rate of change, *p. 462*

Core Concepts

Section 8.4

Even and Odd Functions, *p. 442*
Graphing $f(x) = a(x - h)^2$, *p. 443*
Graphing $f(x) = a(x - h)^2 + k$, *p. 444*
Writing Quadratic Functions of the Form $f(x) = a(x - h)^2 + k$, *p. 445*

Section 8.5

Graphing $f(x) = a(x - p)(x - q)$, *p. 450*
Factors and Zeros, *p. 452*
Using Characteristics to Graph and Write Quadratic Functions, *p. 452*
Using Characteristics to Graph and Write Cubic Functions, *p. 453*

Section 8.6

Linear, Exponential, and Quadratic Functions, *p. 460*
Differences and Ratios of Functions, *p. 461*
Writing Functions to Model Data, *p. 462*
Comparing Functions Using Average Rates of Change, *p. 463*

Mathematical Practices

1. How can you use technology to confirm your answer in Exercise 64 on page 448?

2. How did you use the structure of the equation in Exercise 85 on page 457 to solve the problem?

3. Describe why your answer makes sense considering the context of the data in Exercise 20 on page 466.

Performance Task

Asteroid Aim

Apps take a long time to design and program. One app in development is a game in which players shoot lasers at asteroids. They score points based on the number of hits per shot. The designer wants your feedback. Do you think students will like the game and want to play it? What changes would improve it?

To explore the answers to this question and more, go to ***BigIdeasMath.com***.

8 Chapter Review

Dynamic Solutions available at *BigIdeasMath.com*

8.1 Graphing $f(x) = ax^2$ *(pp. 419–424)*

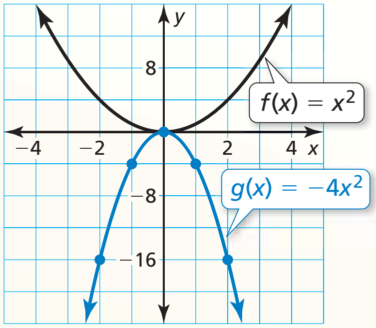
Graph $g(x) = -4x^2$. Compare the graph to the graph of $f(x) = x^2$.

Step 1 Make a table of values.

x	-2	-1	0	1	2
$g(x)$	-16	-4	0	-4	-16

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



▶ The graphs have the same vertex, (0, 0), and the same axis of symmetry, $x = 0$, but the graph of g opens down and is narrower than the graph of f. So, the graph of g is a vertical stretch by a factor of 4 and a reflection in the x-axis of the graph of f.

Graph the function. Compare the graph to the graph of $f(x) = x^2$.

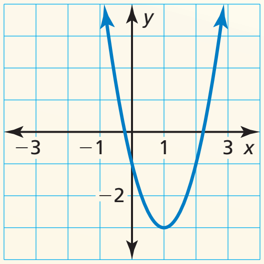
1. $p(x) = 7x^2$

2. $q(x) = \frac{1}{2}x^2$

3. $g(x) = -\frac{3}{4}x^2$

4. $h(x) = -6x^2$

5. Identify characteristics of the quadratic function and its graph.



8.2 Graphing $f(x) = ax^2 + c$ *(pp. 425–430)*

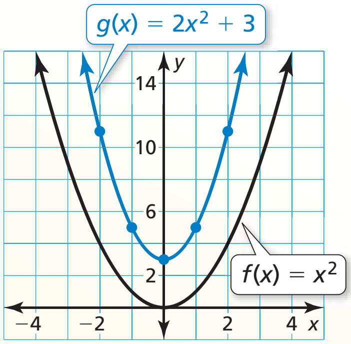
Graph $g(x) = 2x^2 + 3$. Compare the graph to the graph of $f(x) = x^2$.

Step 1 Make a table of values.

x	-2	-1	0	1	2
$g(x)$	11	5	3	5	11

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



▶ Both graphs open up and have the same axis of symmetry, $x = 0$. The graph of g is narrower, and its vertex, (0, 3), is above the vertex of the graph of f, (0, 0). So, the graph of g is a vertical stretch by a factor of 2 and a vertical translation 3 units up of the graph of f.

Graph the function. Compare the graph to the graph of $f(x) = x^2$.

6. $g(x) = x^2 + 5$

7. $h(x) = -x^2 - 4$

8. $m(x) = -2x^2 + 6$

9. $n(x) = \frac{1}{3}x^2 - 5$

8.3 Graphing $f(x) = ax^2 + bx + c$ *(pp. 431–438)*

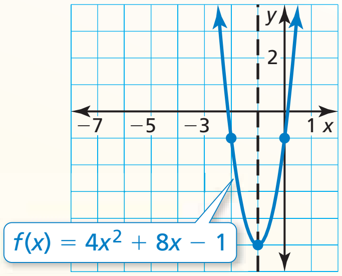
Graph $f(x) = 4x^2 + 8x - 1$. Describe the domain and range.

Step 1 Find and graph the axis of symmetry: $x = -\frac{b}{2a} = -\frac{8}{2(4)} = -1$.

Step 2 Find and plot the vertex. The axis of symmetry is $x = -1$. So, the x-coordinate of the vertex is -1. The y-coordinate of the vertex is $f(-1) = 4(-1)^2 + 8(-1) - 1 = -5$. So, the vertex is $(-1, -5)$.

Step 3 Use the y-intercept to find two more points on the graph. Because $c = -1$, the y-intercept is -1. So, $(0, -1)$ lies on the graph. Because the axis of symmetry is $x = -1$, the point $(-2, -1)$ also lies on the graph.

Step 4 Draw a smooth curve through the points.



▶ The domain is all real numbers. The range is $y \geq -5$.

Graph the function. Describe the domain and range.

10. $y = x^2 - 2x + 7$

11. $f(x) = -3x^2 + 3x - 4$

12. $y = \frac{1}{2}x^2 - 6x + 10$

13. The function $f(t) = -16t^2 + 88t + 12$ represents the height (in feet) of a pumpkin t seconds after it is launched from a catapult. When does the pumpkin reach its maximum height? What is the maximum height of the pumpkin?

8.4 Graphing $f(x) = a(x - h)^2 + k$ *(pp. 441–448)*

Determine whether $f(x) = 2x^2 + 4$ is *even*, *odd*, or *neither*.

$f(x) = 2x^2 + 4$ Write the original function.

$f(-x) = 2(-x)^2 + 4$ Substitute $-x$ for x.

$= 2x^2 + 4$ Simplify.

$= f(x)$ Substitute $f(x)$ for $2x^2 + 4$.

▶ Because $f(-x) = f(x)$, the function is even.

Determine whether the function is *even*, *odd*, or *neither*.

14. $w(x) = 5^x$

15. $r(x) = -8x$

16. $h(x) = 3x^2 - 2x$

Graph the function. Compare the graph to the graph of $f(x) = x^2$.

17. $h(x) = 2(x - 4)^2$

18. $g(x) = \frac{1}{2}(x - 1)^2 + 1$

19. $q(x) = -(x + 4)^2 + 7$

20. Consider the function $g(x) = -3(x + 2)^2 - 4$. Graph $h(x) = g(x - 1)$.

21. Write a quadratic function whose graph has a vertex of $(3, 2)$ and passes through the point $(4, 7)$.

8.5 Using Intercept Form *(pp. 449–458)*

Use zeros to graph $h(x) = x^2 - 7x + 6$.

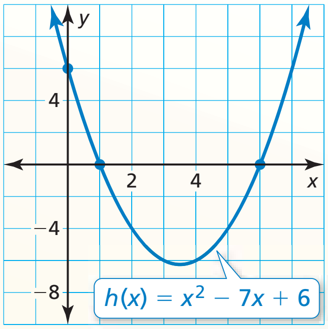
The function is in standard form. The parabola opens up ($a > 0$), and the y-intercept is 6. So, plot (0, 6).

The polynomial that defines the function is factorable. So, write the function in intercept form and identify the zeros.

$h(x) = x^2 - 7x + 6$ Write the function.

$= (x - 6)(x - 1)$ Factor the trinomial.

The zeros of the function are 1 and 6. So, plot (1, 0) and (6, 0). Draw a parabola through the points.



Graph the quadratic function. Label the vertex, axis of symmetry, and *x*-intercepts. Describe the domain and range of the function.

22. $y = (x - 4)(x + 2)$

23. $f(x) = -3(x + 3)(x + 1)$

24. $y = x^2 - 8x + 15$

Use zeros to graph the function.

25. $y = -2x^2 + 6x + 8$

26. $f(x) = x^2 + x - 2$

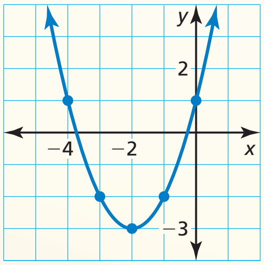
27. $f(x) = 2x^3 - 18x$

28. Write a quadratic function in standard form whose graph passes through (4, 0) and (6, 0).

8.6 Comparing Linear, Exponential, and Quadratic Functions *(pp. 459–468)*

Tell whether the data represent a *linear*, an *exponential*, or a *quadratic* function.

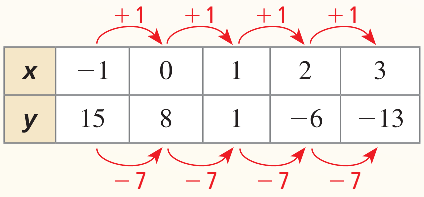
a. (−4, 1), (−3, −2), (−2, −3)
(−1, −2), (0, 1)



▶ The points appear to represent a quadratic function.

b.

x	−1	0	1	2	3
y	15	8	1	−6	−13



x	−1	0	1	2	3
y	15	8	1	−6	−13

▶ The first differences are constant. So, the table represents a linear function.

29. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

x	−1	0	1	2	3
y	512	128	32	8	2

30. The balance y (in dollars) of your savings account after t years is represented by $y = 200(1.1)^t$. The beginning balance of your friend's account is \$250, and the balance increases by \$20 each year. (a) Compare the account balances by calculating and interpreting the average rates of change from $t = 2$ to $t = 7$. (b) Predict which account will have a greater balance after 10 years. Explain.

8 Chapter Test

Graph the function. Compare the graph to the graph of $f(x) = x^2$.

1. $h(x) = 2x^2 - 3$

2. $g(x) = -\frac{1}{2}x^2$

3. $p(x) = \frac{1}{2}(x + 1)^2 - 1$

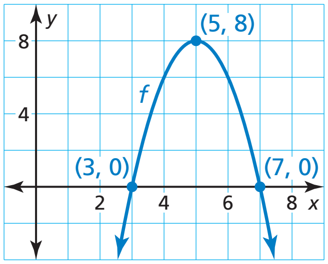
4. Consider the graph of the function *f*.

a. Find the domain, range, and zeros of the function.

b. Write the function *f* in standard form.

c. Compare the graph of *f* to the graph of $g(x) = x^2$.

d. Graph $h(x) = f(x - 6)$.



Use zeros to graph the function. Describe the domain and range of the function.

5. $f(x) = 2x^2 - 8x + 8$

6. $y = -(x + 5)(x - 1)$

7. $h(x) = 16x^2 - 4$

Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Explain your reasoning. Then write the function.

8.

x	−1	0	1	2	3
y	4	8	16	32	64

9.

x	−2	−1	0	1	2
y	−8	−2	0	−2	−8

Write a quadratic function in standard form whose graph satisfies the given conditions. Explain the process you used.

10. passes through (−8, 0), (−2, 0), and (−6, 4)

11. passes through (0, 0), (10, 0), and (9, −27)

12. is even and has a range of $y \geq 3$

13. passes through (4, 0) and (1, 9)

14. The table shows the distances *d* (in miles) that Earth moves in its orbit around the Sun after *t* seconds. Let the time *t* be the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain. Then write a function that models the data.

Time, *t*	1	2	3	4	5
Distance, *d*	19	38	57	76	95

15. You are playing tennis with a friend. The path of the tennis ball after you return a serve can be modeled by the function $y = -0.005x^2 + 0.17x + 3$, where *x* is the horizontal distance (in feet) from where you hit the ball and *y* is the height (in feet) of the ball.

a. What is the maximum height of the tennis ball?

b. You are standing 30 feet from the net, which is 3 feet high. Will the ball clear the net? Explain your reasoning.

16. Find values of *a*, *b*, and *c* so that the function $f(x) = ax^2 + bx + c$ is (a) even, (b) odd, and (c) neither even nor odd.

17. Consider the function $f(x) = x^2 + 4$. Find the average rate of change from $x = 0$ to $x = 1$, from $x = 1$ to $x = 2$, and from $x = 2$ to $x = 3$. What do you notice about the average rates of change when the function is increasing?

8 Cumulative Assessment

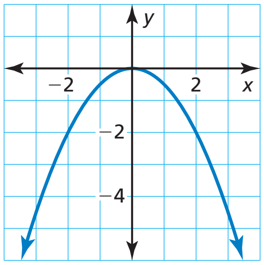
1. Which function is represented by the graph?

Ⓐ $y = \frac{1}{2}x^2$

Ⓑ $y = 2x^2$

Ⓒ $y = -\frac{1}{2}x^2$

Ⓓ $y = -2x^2$



2. Find all numbers between 0 and 100 that are in the range of the function defined below.

$$f(1) = 1,\ f(2) = 1,\ f(n) = f(n - 1) + f(n - 2)$$

3. The function $f(t) = -16t^2 + v_0t + s_0$ represents the height (in feet) of a ball t seconds after it is thrown from an initial height s_0 (in feet) with an initial vertical velocity v_0 (in feet per second). The ball reaches its maximum height after $\frac{7}{8}$ second when it is thrown with an initial vertical velocity of ______ feet per second.

4. Classify each system of equations by the number of solutions.

$y = 6x + 9$
$y = -\frac{1}{6}x + 9$

$7x + 4y = 12$
$8y - 12 = -14x$

$2x + 4y = -2$
$10x + 4y = -2$

$3x + y = 5$
$-15 + 3y + 9x = 0$

$y - 2x = \frac{3}{2}$
$-3 + 2y = 4x$

$y = -3x + 5$
$y = -3x + 9$

5. Your friend claims that quadratic functions can have two, one, or no real zeros. Do you support your friend's claim? Use graphs to justify your answer.

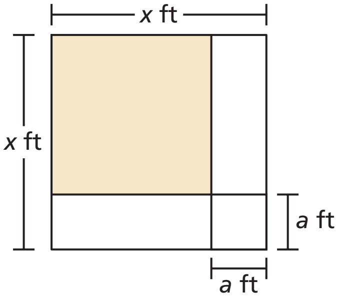
6. Which polynomial represents the area (in square feet) of the shaded region of the figure?

Ⓐ $a^2 - x^2$

Ⓑ $x^2 - a^2$

Ⓒ $x^2 - 2ax + a^2$

Ⓓ $x^2 + 2ax + a^2$



7. Consider the functions represented by the tables.

x	0	1	2	3
p(x)	−4	−16	−28	−40

x	1	2	3	4
r(x)	0	15	40	75

x	1	2	3	4
s(x)	72	36	18	9

x	1	3	5	7
t(x)	3	−5	−21	−45

a. Classify each function as *linear*, *exponential*, or *quadratic*.

b. Order the functions from least to greatest according to the average rates of change between $x = 1$ and $x = 3$.

8. Complete each function using the symbols + or −, so that the graph of the quadratic function satisfies the given conditions.

a. $f(x) = 5(x \square 3)^2 \square 4$; vertex: $(-3, 4)$

b. $g(x) = -(x \square 2)(x \square 8)$; x-intercepts: −8 and 2

c. $h(x) = \square 3x^2 \square 6$; range: $y \geq -6$

d. $j(x) = \square 4(x \square 1)(x \square 1)$; range: $y \leq 4$

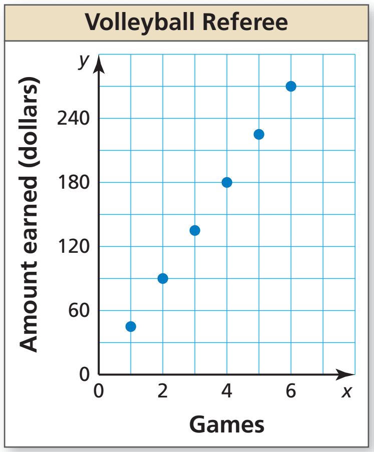
9. The graph shows the amounts y (in dollars) that a referee earns for refereeing x high school volleyball games.

a. Does the graph represent a linear or nonlinear function? Explain.

b. Describe the domain of the function. Is the domain discrete or continuous?

c. Write a function that models the data.

d. Can the referee earn exactly $500? Explain.



10. Which expressions are equivalent to $(b^{-5})^{-4}$?

- b^{-20}
- $b^{-6}b^{-14}$
- $(b^{-4})^{-5}$
- b^{-9}
- $(b^{-2})^{-7}$
- $b^{-5}b^{-4}$
- $(b^{10})^2$
- b^{20}
- $(b^{-12})^3$
- $b^{12}b^8$